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FRACTIONAL POISSON EQUATIONS AND ERGODIC THEOREMS FOR FRACTIONAL COBOUNDARIES

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Dedicated to Antoine Brunel on his eightieth birthday

ABSTRACT

For a given contraction T in a Banach space X and $0 < \alpha < 1$, we define the contraction $T_\alpha = \sum_{j=1}^{\infty} a_j T^j$, where $\{a_j\}$ are the coefficients in the power series expansion $(1-t)^\alpha = 1 - \sum_{j=1}^{\infty} a_j t^j$ in the open unit disk, which satisfy $a_j > 0$ and $\sum_{j=1}^{\infty} a_j = 1$. The operator calculus justifies the notation $(I-T)^\alpha := I - T_\alpha$ (e.g., $(I - T_{1/2})^2 = I - T$). A vector $y \in X$ is called an **α-fractional coboundary for T** if there is an $x \in X$ such that $(I-T)^\alpha x = y$, i.e., y is a coboundary for T_α. The fractional Poisson equation for T is the Poisson equation for T_α. We show that if $(I-T)X$ is not closed, then $(I-T)^\alpha X$ strictly contains $(I-T)X$ (but has the same closure).

For T mean ergodic, we obtain a series solution (converging in norm) to the fractional Poisson equation. We prove that $y \in X$ is an α-fractional coboundary if and only if $\sum_{k=1}^{\infty} T^k y / k^{1-\alpha}$ converges in norm, and conclude that $\lim_n \|(1/n^{1-\alpha}) \sum_{k=1}^{n} T^k y\| = 0$ for such y.

For a Dunford–Schwartz operator T on L_1 of a probability space, we consider also a.e. convergence. We prove that if $f \in (I-T)^\alpha L_1$ for some $0 < \alpha < 1$, then the one-sided Hilbert transform $\sum_{k=1}^{\infty} T^k f/k$ converges a.e. For $1 < p < \infty$, we prove that if $f \in (I-T)^\alpha L_p$ with $\alpha > 1 - 1/p = 1/q$, then $\sum_{k=1}^{\infty} T^k f / k^{1/p}$ converges a.e., and thus $(1/n^{1/p}) \sum_{k=1}^{n} T^k f$ converges a.e. to zero. When $f \in (I-T)^{1/q} L_p$ (the case $\alpha = 1/q$), we prove that $(1/n^{1/p} (\log n)^{1/q}) \sum_{k=1}^{n} T^k f$ converges a.e. to zero.

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1. Introduction

The ergodic theorem does not contain any general speed of convergence. In fact, there is no universal speed of convergence in Birkhoff's pointwise ergodic theorem for probability preserving transformations (see the discussion in [Kr, p. 14]). However, in many situations of ergodic theory or probability, certain speeds of convergence are essential, and one is interested in obtaining them from assumptions that link the integrable function f to the transformation (or, more generally, to the Markov operator) under consideration. This is the main theme of the present work.

It is well-known that the ergodic theorem (the mean ergodic theorem for a contraction T on a reflexive Banach space X, or the pointwise ergodic theorem for T induced by a measure preserving transformation) is based upon the ergodic decomposition of the space $X = \{x : Tx = x\} \oplus \overline{(I-T)X}$. The elements of $(I-T)X$ are called **coboundaries**. The characterization *y is a coboundary if and only if* $\sup_n \|\sum_{k=0}^{n-1} T^k y\| < \infty$ (obtained in [Le] for certain unitary operators on a Hilbert space, in [BuW] for contractions in reflexive spaces, and in [LiSi] for contractions in L_1) is a characterization by the rate of convergence of $1/n$ in the mean ergodic theorem. For a coboundary y, the Poisson equation $y = (I-T)x$ can be solved by using the averages of the sequence $\{\sum_{k=0}^{n-1} T^k y\}$ [LiSi]. For an ergodic stationary Markov chain with transition probability operator P and invariant initial probability m, Gordin and Lifshits [GoLif-1] obtained a central limit theorem for additive functionals f on the state space which are $L_2(m)$-coboundaries for the Markov operator P.

The starting point of this work was the result obtained by Kipnis and Varadhan [KiV] for reversible ergodic Markov chains with an invariant probability (i.e., P is self-adjoint in $L_2(m)$): the central limit theorem holds for functionals f which are of the form $f = (I-P)^{\frac{1}{2}} g$ with $g \in L_2(m)$. When (unaware of the result in [BoI, §IV.7], announced in [GoLif-2]) we extended the above result to P normal [DeLi], we saw a connection with the operator $I - \sqrt{I-T}$, used by Brunel [Br] in the context of multiparameter pointwise ergodic theorems. This inspired the definition of $(I-T)^\alpha$ for any $\alpha \in (0,1)$, when T is a contraction on a Banach space. For T mean ergodic, we obtain a characterization of the elements of the image of $(I-T)^\alpha$ (called α-fractional coboundaries), and a series solution of the fractional Poisson equation $(I-T)^\alpha x = y$. In section 2 we consider the problem for a contraction in a general Banach space, and our results are expressed in terms of the norm or the weak topology of the space. In section 3 we deal with probability preserving transformations and general Dunford–Schwartz operators, and the problems are treated from the point of view of a.e. convergence. Special results for normal contractions in a Hilbert space are given in section 4.

2. Fractional Poisson equations in Banach spaces

Let T be a contraction (bounded linear operator of norm ≤ 1) in a Banach space X. In this section we define bounded linear operators $(I-T)^\alpha$ for $0<\alpha<1$. For $r>0$ real, we define the operators $(I-T)^r := (I-T)^{[r]}(I-T)^{\{r\}}$ (where $[r]$ and $\{r\}$ are the integral and fractional parts of r), and obtain the semigroup property $(I-T)^{r+s} = (I-T)^r(I-T)^s$. For $0<\alpha<1$ we then study the associated **fractional Poisson equation** $(I-T)^\alpha x = y$ for a given $y \in X$.

For a fixed $\alpha \in (0,1)$, we have the power series expansion $(1-t)^\alpha = 1 - \sum_{j=1}^\infty a_j^{(\alpha)} t^j$, where $a_1^{(\alpha)} = \alpha$, and

$$a_j^{(\alpha)} = \frac{\alpha(1-\alpha)\cdots(j-1-\alpha)}{j!} \quad \text{for } j \geq 2. \tag{2.1}$$

The asymptotic behaviour of $a_j^{(\alpha)}$ is given in [Z, vol. I p. 77]:

$$\frac{n+1}{\alpha} a_{n+1}^{(\alpha)} = \frac{n^{-\alpha}}{\Gamma(1-\alpha)}[1+O(1/n)]. \tag{2.2}$$

We have $a_j^{(\alpha)} > 0$ for $j \geq 1$ and $\sum_{j=1}^\infty a_j^{(\alpha)} = 1$, so for a contraction T in a Banach space the series $\sum_{j=1}^\infty a_j^{(\alpha)} T^j$ is absolutely convergent in the operator norm, and defines a contraction T_α.

Definition: For a contraction T in a Banach space X and $0<\alpha<1$, we define $(I-T)^\alpha = I - T_\alpha = I - \sum_{j=1}^\infty a_j^{(\alpha)} T^j$.

Remark: The referee has noted that the preceding definition is a particular case of the general definition of the fractional power of the inifinitesimal generator A of a strongly continuous semigroup of operators $\{T_t : t \geq 0\}$. In this general situation we define

$$(-A)^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda I - A)^{-1}(-Ax)d\lambda \quad \forall x \in \mathcal{D}(A).$$

Then $\widehat{A}_\alpha = -(-A)^\alpha$ is the infinitesimal generator of the semigroup $\{\widehat{T}_t : t \geq 0\}$ subordinated to $\{T_t\}$ by the semigroup of stable probability distributions of index α on $\mathbf{R}^+$ [Y, §IX.11]. In our context, with T a contraction and $A = T - I$, insertion of the Neumann series for $(\lambda I - A)^{-1} = [(\lambda+1)I - T]^{-1}$ in the preceding integral will yield the power series expansion for $(I-T)^\alpha$ given in our definition.

Let $0 < \alpha, \beta < 1$. When $\alpha + \beta \leq 1$, we multiply the power series $(1-t)^\alpha(1-t)^\beta = (1-t)^{\alpha+\beta}$ according to Cauchy's rule, and easily conclude that $(I-T)^\alpha(I-T)^\beta = (I-T)^{\alpha+\beta}$. When $\alpha+\beta > 1$, we put $\gamma = \alpha+\beta-1$, and multiplying $(1-t)^\alpha(1-t)^\beta = (1-t)^\gamma(1-t)$ we obtain $(I-T)^\alpha(I-T)^\beta = (I-T)^\gamma(I-T)$. It is now immediate that $(I-T)^r$ has the semigroup property.

For fixed $0 < \alpha < 1$, we now study the image of $(I-T)^\alpha$. The fractional Poisson equation for T is in fact the Poisson equation for the contraction T_α.

PROPOSITION 2.1: *Let T be a contraction in a Banach space X, and let $0 < \alpha < 1$.*

(i) *$(I-T)^\beta X \subset (I-T)^\alpha X$ for $\alpha < \beta \leq 1$.*

(ii) *$\overline{(I-T)^\alpha X} = \overline{(I-T)X}$.*

(iii) *$(I-T)^\alpha X$ is closed if and only if $(I-T)X$ is closed.*

(iv) *$(I-T)^\alpha x = 0$ if and only if $(I-T)x = 0$ (i.e., T and T_α have the same fixed points).*

Proof: (i) follows from $(I-T)^\beta = (I-T)^\alpha(I-T)^{\beta-\alpha}$.

(ii) Let $y = (I-T)^\alpha x$, so $y = x - \sum_{j=1}^\infty a_j^{(\alpha)} T^j x = \sum_{j=1}^\infty a_j^{(\alpha)}(I-T^j)x$. Since $(I-T^j)x = (I-T)\sum_{k=0}^{j-1} T^k x$, we have $y \in \overline{(I-T)X}$. This and (i) (with $\beta = 1$) yield

$$\overline{(I-T)X} \subset \overline{(I-T)^\alpha X} \subset \overline{(I-T)X}.$$

(iii) By (i) and (ii), $(I-T)X \subset (I-T)^\alpha X \subset \overline{(I-T)X}$, so $(I-T)X$ closed implies $(I-T)^\alpha X$ closed. Assume now that $(I-T)^\alpha X$ is closed. Denote $Y = \overline{(I-T)X}$, so by (ii), $I - T_\alpha$ has closed range Y. The closed subspace Y is T-invariant, hence also T_α-invariant. Clearly T_α has no fixed points in Y, so $I - T_\alpha$ is invertible on Y with continuous inverse. Assume first $\alpha \geq \frac{1}{2}$. Then $(I-T)^{\frac{1}{2}}(I-T)^{\alpha-\frac{1}{2}} = (I-T)^\alpha$, so invertibility of $(I-T)^\alpha$ on Y implies invertibility of $(I-T)^{\frac{1}{2}}$ on Y, so $(I-T)Y = [(I-T)^{\frac{1}{2}}]^2 Y = Y$, and $Y \subset (I-T)X$, so equality holds. Assume now $0 < \alpha < \frac{1}{2}$. Let K be the first positive integer with $K\alpha \geq \frac{1}{2}$ (and then $K\alpha < 1$). The invertibility on Y of $(I-T)^\alpha$ implies that of $(I-T)^{K\alpha} = [(I-T)^\alpha]^K$, and we apply to $K\alpha$ the first part of the proof.

(iv) follows from the Brunel–Falkowitz lemma [Kr]. ∎

PROPOSITION 2.2: *Let T be a contraction in a Banach space X. If $(I-T)X$ is not closed, then for $0 < \alpha < \beta \leq 1$ we have $(I-T)^\beta X \neq (I-T)^\alpha X$.*

Proof: By Proposition 2.1(iii), the space $(I-T)^{\beta-\alpha}X$ is not closed, but its closure is $Y = \overline{(I-T)X}$, by Proposition 2.1(ii). Hence there is $x \in Y$ which is not in $(I-T)^{\beta-\alpha}X$, and let $y=(I-T)^\alpha x$. If we assume that $(I-T)^\alpha X=(I-T)^\beta X$, then there is z with $y=(I-T)^\beta z=(I-T)^\alpha(I-T)^{\beta-\alpha}z$ (since $\beta>\alpha$). Hence $(I-T)^\alpha[x-(I-T)^{\beta-\alpha}z]=0$. By Proposition 2.1(ii), $(I-T)^{\beta-\alpha}z\in Y$, so we have that $x-(I-T)^{\beta-\alpha}z$ is a fixed point of T_α in Y, so by Proposition 2.1(iv) it is 0, which yields $x\in(I-T)^{\beta-\alpha}X$ — a contradiction to the choice of x. ∎

Remark: When $(I-T)X$ is closed, the operator T is uniformly ergodic [Li], and $I-T$ is invertible on its image. This is a "rare" case (operators induced by measure preserving transformations are not uniformly ergodic).

For a contraction T in X and $0<\alpha<1$, we want to study $(I-T)^\alpha X$. A vector $y\in X$ is called an α-**fractional coboundary** for T if it is in $(I-T)^\alpha X$, i.e., the fractional Poisson equation $(I-T)^\alpha x=y$ has a solution (which means that y is a coboundary for T_α). Formally, solving the fractional Poisson equation means inverting $(I-T)^\alpha$ (on its image). Using term by term differentiation of $(1-t)^\beta=1-\sum_{j=1}^\infty a_j^{(\beta)}t^j$ for $|t|<1$, and applying it to $\beta=1-\alpha$, we find

$$[(1-t)^\alpha]^{-1}=\frac{1}{1-\alpha}\sum_{j=1}^\infty ja_j^{(1-\alpha)}t^{j-1}=\frac{1}{1-\alpha}\sum_{j=0}^\infty (j+1)a_{j+1}^{(1-\alpha)}t^j\quad\text{for } |t|<1.$$

At $t=1$ the series diverges, but we still want to use it for the inversion of $(I-T)^\alpha$. Now the question is: if $y\in(I-T)^\alpha X$, does the series $\frac{1}{1-\alpha}\sum_{j=0}^N(j+1)a_{j+1}^{(1-\alpha)}T^jy$ converge (weakly or strongly), as $N\to\infty$, to a solution x of $(I-T)^\alpha x=y$? Note that since y is in $Y=\overline{(I-T)X}$, also the limit of the series, if it exists, must be in Y.

For $|t|<1$ we multiply the power series

$$(2.3)\quad 1-\alpha=(1-\alpha)(1-t)^\alpha[(1-t)^\alpha]^{-1}=(1-\sum_{j=1}^\infty a_j^{(\alpha)}t^j)[\sum_{k=0}^\infty(k+1)a_{k+1}^{(1-\alpha)}t^k]$$

and computing the coefficients of t^n according to Cauchy's rule we obtain

$$(2.4)\qquad (n+1)a_{n+1}^{(1-\alpha)}=\sum_{k=0}^{n-1}(k+1)a_{k+1}^{(1-\alpha)}a_{n-k}^{(\alpha)}=\sum_{j=1}^{n}(n+1-j)a_{n+1-j}^{(1-\alpha)}a_j^{(\alpha)}.$$

LEMMA 2.3: *For $N\geq 0$ we have*

$$(2.5)\qquad \sum_{k=N+1}^\infty\sum_{j=0}^N(j+1)a_{j+1}^{(1-\alpha)}a_{k-j}^{(\alpha)}=1-\alpha.$$

Proof: We need to prove only for $N > 0$. For $0 < t < 1$ we obtain, from the absolute convergence, using (2.4) for the last equality,

$$\begin{aligned}0 <& \left(1 - \sum_{k=1}^{\infty} a_k^{(\alpha)} t^k\right) \left[\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} t^j\right] = a_1^{(1-\alpha)} + \sum_{j=1}^{N} (j+1) a_{j+1}^{(1-\alpha)} t^j \\ &- \sum_{n=1}^{N} \left(\sum_{k=1}^{n} a_k^{(\alpha)} (n+1-k) a_{n+1-k}^{(1-\alpha)}\right) t^n - \sum_{n=N+1}^{\infty} \left(\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} a_{n-j}^{(\alpha)}\right) t^n \\ =& 1 - \alpha - \sum_{n=N+1}^{\infty} \left(\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} a_{n-j}^{(\alpha)}\right) t^n.\end{aligned}$$

Hence for $N > 0$ fixed, we have $\sum_{n=N+1}^{\infty} (\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} a_{n-j}^{(\alpha)}) t^n \le 1 - \alpha$ for every $0 < t < 1$. Letting t increase to 1, we obtain that the left hand side of (2.5) converges to $1 - \alpha$ from below. ∎

PROPOSITION 2.4: *Let T be a contraction in Banach space X and $0 < \alpha < 1$. Then*

(i) *for every $z \in X$ we have*

$$\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} T^j (I-T)^\alpha z = (1-\alpha) z - \sum_{n=N+1}^{\infty} \left(\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} a_{n-j}^{(\alpha)}\right) T^n z;$$

(ii) $\sup_N \left\| \sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} T^j (I-T)^\alpha \right\| \le 2(1-\alpha)$.

Proof: (i) Because of the absolute convergence in defining T_α, we have the following, with strong convergence:

$$\begin{aligned}\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} T^j (I-T)^\alpha z =& \sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} T^j \left(z - \sum_{k=1}^{\infty} a_k^{(\alpha)} T^k z\right) \\ =& a_1^{(1-\alpha)} z + \sum_{j=1}^{N} (j+1) a_{j+1}^{(1-\alpha)} T^j z - \sum_{j=0}^{N} \sum_{k=1}^{\infty} (j+1) a_{j+1}^{(1-\alpha)} a_k^{(\alpha)} T^{j+k} z \\ =& (1-\alpha) z + \sum_{n=1}^{N} \left[(n+1) a_{n+1}^{(1-\alpha)} - \sum_{k=1}^{n} (n+1-k) a_{n+1-k}^{(1-\alpha)} a_k^{(\alpha)}\right] T^n z \\ &- \sum_{n=N+1}^{\infty} \left[\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} a_{n-j}^{(\alpha)}\right] T^n z \\ =& (1-\alpha) z - \sum_{n=N+1}^{\infty} \left[\sum_{j=0}^{N} (j+1) a_{j+1}^{(1-\alpha)} a_{n-j}^{(\alpha)}\right] T^n z\end{aligned}$$

by using (2.4) for the last equality.

(ii) By Lemma 2.3 the operator $R_N^{(\alpha)} := \sum_{k=N+1}^{\infty}\left[\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}a_{k-j}^{(\alpha)}\right]T^k$ is well defined, with operator norm convergence, and $\|R_N^{(\alpha)}\| \le 1-\alpha$ for every N. With the computations in the proof of (i), this yields

$$\left\|\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}T^j(I-T)^\alpha z\right\| \le \|(1-\alpha)z\| + \|R_N^{(\alpha)}z\| \le 2(1-\alpha)\|z\|.$$

LEMMA 2.5: *For $0<\alpha<1$ and $k\ge 1$ we have $\sum_{j=k}^{\infty} a_j^{(\alpha)} = (1/\alpha)ka_k^{(\alpha)}$.*

Proof: For $|t|<1$ we compute the absolutely convergent series

$$\sum_{k=1}^{\infty}\Big(\sum_{j=k}^{\infty} a_j^{(\alpha)}\Big)t^k(1-t) = t - \sum_{k=1}^{\infty} a_k t^{k+1} = t(1-t)^\alpha.$$

Hence

$$\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty} a_j^{(\alpha)}\right)t^k = t(1-t)^{\alpha-1} = -\frac{t}{\alpha}[(1-t)^\alpha]' = \frac{1}{\alpha}\sum_{k=1}^{\infty} ka_k^{(\alpha)}t^k$$

and the result follows by equating the coefficients on both sides. ∎

PROPOSITION 2.6: *let T be a contraction in a Banach space X, and $0<\alpha<1$. Then*

$$\lim_{N\to\infty}\left\|\sum_{k=N+1}^{\infty}\left[\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}a_{k-j}^{(\alpha)}\right]T^k(I-T)\right\| = 0.$$

Proof: By Lemma 2.3, the operators are well defined, with operator norm convergence of the series, and for any $x\in X$ we have

$$\sum_{k=N+1}^{\infty}\left[\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}a_{k-j}^{(\alpha)}\right]T^k(I-T)x$$
$$= \sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}\left[a_{N+1-j}^{(\alpha)}T^{N+1}x + \sum_{k=N+2}^{\infty}(a_{k-j}^{(\alpha)} - a_{k-j-1}^{(\alpha)})T^k x\right].$$

By (2.4), the norm of the first term is majorized by $\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}a_{N+1-j}^{(\alpha)}\|x\| = \|x\|(N+2)a_{N+2}^{(1-\alpha)}$. By (2.1), $a_j^{(\alpha)}$ is decreasing (to zero), so the triangle inequality yields a majorization of the second term by

$$\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}\sum_{k=N+2}^{\infty}(a_{k-j-1}^{(\alpha)} - a_{k-j}^{(\alpha)})\|x\| = \|x\|(N+2)a_{N+2}^{(1-\alpha)}.$$

Hence, using Lemma 2.5, $\|\sum_{k=N+1}^{\infty}\left[\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}a_{k-j}^{(\alpha)}\right]T^k(I-T)\| \le 2(N+2)a_{N+2}^{(1-\alpha)} = 2(1-\alpha)\sum_{j=N+2}^{\infty}a_j^{(1-\alpha)} \to 0.$ ∎

THEOREM 2.7: *Let T be a contraction in a Banach space X, and $0<\alpha<1$. If $y=(I-T)^\alpha x$ with $x\in\overline{(I-T)X}$, then*

$$\lim_{N\to\infty}\|\frac{1}{1-\alpha}\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}T^jy - x\| = 0.$$

Proof: With $R_N^{(\alpha)}$ as defined in Proposition 2.4, that proposition yields

$$\begin{aligned}\frac{1}{1-\alpha}\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}T^jy &= \frac{1}{1-\alpha}\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}T^j(I-T)^\alpha x\\ &= x-\frac{1}{1-\alpha}R_N^{(\alpha)}x.\end{aligned}$$

By Lemma 2.3, $\|R_N^{(\alpha)}\|\le 1-\alpha$ for every N, and $\|R_N^{(\alpha)}(I-T)\|\to 0$ by Proposition 2.6, so $\|R_N^{(\alpha)}x\|\to 0$ by the assumption $x\in\overline{(I-T)X}$. ∎

Recall that a contraction is called **mean ergodic** if $Ex := \lim_n \frac{1}{n}\sum_{k=1}^{n}T^kx$ exists for every $x\in X$. The limit operator E is then a projection onto the space $F(T) := \{y\in X : Ty = y\}$, with null space $\overline{(I-T)X}$. The ergodic decomposition $X = F(T)\oplus\overline{(I-T)X}$ is equivalent to mean ergodicity of T (see [Kr] for more information).

By Proposition 2.1, mean ergodicity of T is equivalent to that of T_α (for any fixed $0<\alpha<1$). When T is mean ergodic, the fractional Poisson equation has a solution in X if and only if it has one in $Y=\overline{(I-T)X}$; the solution in Y is unique, and Theorem 2.7 yields a series representation for this solution.

COROLLARY 2.8: *Let T be a mean ergodic contraction in a Banach space X, and $0<\alpha<1$. If $y\in(I-T)^\alpha X$, then the series $\frac{1}{1-\alpha}\sum_{j=0}^{\infty}(j+1)a_{j+1}^{(1-\alpha)}T^jy$ converges strongly, its limit x is in $\overline{(I-T)X}$, and satisfies $(I-T)^\alpha x = y$.*

For any contraction T, the space of $x\in X$ for which $\lim_n\frac{1}{n}\sum_{k=1}^{n}T^kx$ exists, called the **ergodic subspace** of T, is a closed invariant subspace, which equals $F(T)\oplus\overline{(I-T)X}$, and the restriction of T to it is mean ergodic. Since every fractional coboundary is in the ergodic subspace, Corollary 2.8 allows us to solve the fractional Poisson equation when there is a solution in the ergodic subspace (and only in that case, since the series converges to an element of $\overline{(I-T)X}$).

LEMMA 2.9: *Let T be a contraction in a Banach space X, $y \in X$, and $0 < \alpha < 1$. If for some subsequence $\{N_k\}$ we have $\sup_{N_k} \|\sum_{j=0}^{N_k}(j+1)a_{j+1}^{(1-\alpha)}T^j y\| = K < \infty$, then $y \in \overline{(I-T)X} = \overline{(I-T)^\alpha X}$.*

Proof: Take $x^* \in X^*$ satisfying $T^*x^* = x^*$. Then for every N_k we have

$$|\langle x^*, x\rangle| \sum_{j=0}^{N_k}(j+1)a_{j+1}^{(1-\alpha)} = \left|\sum_{j=0}^{N_k}(j+1)a_{j+1}^{(1-\alpha)}\langle x^*, T^j x\rangle\right| \leq K\|x^*\|.$$

Since $\sum_{j=1}^{\infty} j a_j^{(1-\alpha)} = \infty$, we must have $\langle x^*, x\rangle = 0$. By the Hahn–Banach theorem $x \in \overline{(I-T)X}$. Proposition 2.1(ii) gives the equality $\overline{(I-T)^\alpha X} = \overline{(I-T)X}$. ∎

PROPOSITION 2.10: *Let T be a contraction in a Banach space X, $y \in X$, and $0 < \alpha < 1$. If some subsequence $(1/(1-\alpha))\sum_{j=0}^{N_k}(j+1)a_{j+1}^{(1-\alpha)}T^j y$ converges weakly to x, then $x \in \overline{(I-T)X}$, $(I-T)^\alpha x = y$, and*

$$\lim_{N\to\infty}\left\|\frac{1}{1-\alpha}\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}T^j y - x\right\| = 0.$$

Proof: By the previous lemma, $y \in \overline{(I-T)X}$, and since a closed subspace is weakly closed, also $x \in \overline{(I-T)X}$. By the weak continuity of $(I-T)^\alpha$ and Proposition 2.4(i), we have

$$(I-T)^\alpha x = \text{w-}\lim_{k\to\infty}\frac{1}{1-\alpha}\sum_{j=0}^{N_k}(j+1)a_{j+1}^{(1-\alpha)}T^j(I-T)^\alpha y = y - \text{w-}\lim\frac{1}{1-\alpha}R_{N_k}^{(\alpha)}y.$$

Since $y \in \overline{(I-T)X}$, Lemma 2.3 and Proposition 2.6 yield $\|R_N^{(\alpha)}y\| \to 0$, and we have $(I-T)^\alpha x = y$. The strong convergence of the whole sequence follows from Theorem 2.7. ∎

THEOREM 2.11: *Let T be a mean ergodic contraction in a Banach space X, and $0 < \alpha < 1$. Then the following conditions are equivalent for $y \in X$:*

(i) *$y \in (I-T)^\alpha X$.*

(ii) *The series $\frac{1}{1-\alpha}\sum_{j=0}^{\infty}(j+1)a_{j+1}^{(1-\alpha)}T^j y$ converges strongly.*

(iii) *A subsequence $\frac{1}{1-\alpha}\sum_{j=0}^{N_k}(j+1)a_{j+1}^{(1-\alpha)}T^j y$ converges weakly.*

(iv) *The series $\sum_{j=1}^{\infty} T^j y/j^{1-\alpha}$ converges strongly.*

When these conditions are satisfied, the sum of the series in (ii) is the unique $x \in \overline{(I-T)X}$ satisfying $(I-T)^\alpha x = y$.

Proof: (i) $\Rightarrow$ (ii) by Corollary 2.8, and (ii) obviously implies (iii). (iii) $\Rightarrow$ (i) by Proposition 2.10.

By (2.2), $\epsilon_j := (j+1)a_{j+1}^{(1-\alpha)} - C_\alpha j^{-(1-\alpha)} = O(1/j^{2-\alpha})$. Hence

$$\sum_{j=1}^{N}(j+1)a_{j+1}^{(1-\alpha)}T^j y = C_\alpha \sum_{j=1}^{N}\frac{T^j y}{j^{1-\alpha}} + \sum_{j=1}^{N}\epsilon_j T^j y. \tag{2.6}$$

Since the last series on the right converges strongly, the series in (ii) and (iv) converge or diverge together. ∎

COROLLARY 2.12: *Let T be a contraction in a reflexive Banach space, and let $0 < \alpha < 1$. For $y \in X$, the equivalent conditions of the previous theorem are all equivalent to each of the following conditions:*

(v) $\sup_N \|\sum_{j=0}^N (j+1)a_{j+1}^{(1-\alpha)}T^j y\| < \infty$.

(vi) $\liminf_N \|\sum_{j=0}^N (j+1)a_{j+1}^{(1-\alpha)}T^j y\| < \infty$.

(vii) $\sup_N \left\|\sum_{j=1}^N \frac{T^j y}{j^{1-\alpha}}\right\| < \infty$.

Proof: Clearly (ii) $\Rightarrow$ (v) $\Rightarrow$ (vi). By reflexivity, (vi) $\Rightarrow$ (iii). The equivalence of (v) and (vii) follows from (2.6). ∎

Remark: For any contraction T on a Banach space X, Proposition 2.4(ii) yields that condition (v) above holds for $y \in (I-T)^\alpha X$. However, condition (v) does not imply $y \in (I-T)^\alpha X$, even for T mean ergodic. Let S be a contraction on a Banach space Z, which is not mean ergodic, and let $z \in Z$ be outside the ergodic subspace of S. Let $y = (I-S)^\alpha z$, and X the closed subspace generated by $\{S^j y\}$. Then X is an S-invariant subspace of the ergodic subspace, and we define T to be the restriction of S to X. Then $(I-T)^\alpha$ is the restriction to X of $(I-S)^\alpha$. If there is $x \in X$ satisfying $(I-T)^\alpha x = y$, then $x - z$ will be S_α-invariant, hence S-invariant, implying z is in the ergodic subspace — a contradiction.

THEOREM 2.13: *Let $X = Z^*$ be the dual of a Banach space Z, let $T = S^*$ be the dual of a contraction S in Z, and let $0 < \alpha < 1$. Then the following conditions are equivalent for $y \in X$:*

(i) $y \in (I-T)^\alpha X$.

(ii) $\sup_N \|\sum_{j=0}^N (j+1)a_{j+1}^{(1-\alpha)}T^j y\| < \infty$.

(iii) $\sup_N \|\sum_{j=1}^N T^j y/j^{1-\alpha}\| < \infty$.

(iv) *There exists $K > 0$ such that $|\langle y, z\rangle| \le K\|(I-S)^\alpha z\|$ for every $z \in Z$.*

Proof: (i) $\Rightarrow$ (ii) follows from Proposition 2.4(ii). (ii) and (iii) are equivalent by (2.6).

The boundedness condition (ii) implies the existence of a weak-* limit point x for the sequence $\{\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}T^j y\}$. By Lemma 2.9, $y \in \overline{(I-T)X}$. By the definitions, $(I-T)^\alpha = [(I-S)^\alpha]^*$, so it is weak-* continuous. The computations of Proposition 2.10, taken this time with weak-* limits, yield $(I-T)^\alpha x = y$.

If $y = (I-T)^\alpha x$, then (iv) holds with $K = \|x\|$. Assume (iv). Then for $z \in Z$ we have, using Proposition 2.4(ii) for the last inequality,

$$\left|\left\langle \sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}T^j y, z\right\rangle\right| = \left|\left\langle y, \sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}S^j z\right\rangle\right|$$
$$\leq K\left\|\sum_{j=0}^{N}(j+1)a_{j+1}^{(1-\alpha)}S^j(I-S)^\alpha z\right\| \leq 2K(1-\alpha)\|z\|.$$

Hence (iv) implies (ii). ∎

Remarks: 1. Although y which satisfies (ii) of the previous theorem is in $\overline{(I-T)X}$ by Lemma 2.9, the weak-* limit point x need not be in $\overline{(I-T)X}$ (though it is in its weak-* closure). For an example, let $Z = \ell_1$, so $X = \ell_\infty$, and take S the shift $S(a_1, a_2, \ldots, a_j, \ldots) = (a_2, a_3, \ldots, a_{j+1}, \ldots)$. Its dual $T = S^*$ is $T(b_1, b_2, b_3, \ldots) = (0, b_1, b_2, \ldots)$. Clearly T has no fixed points. The constant sequence $\mathbf{1} = (1, 1, 1, \ldots)$ is not in $\overline{(I-T)X}$, since $1 = \frac{1}{n}\sum_{k=0}^{n-1}T^k\mathbf{1}(n)$ shows the averages do not converge to 0 in norm. Let $y = (I-T)^\alpha\mathbf{1}$. The only $x \in X$ satisfying $(I-T)^\alpha x = y$ is $\mathbf{1}$ because of the absence of fixed points.

2. Condition (iv) of Theorem 2.13 appears in [KiV] for T symmetric in a Hilbert space and $\alpha = \frac{1}{2}$.

The previous results can yield rates of convergence in the mean ergodic theorem, when we use the following Banach space version of Kronecker's Lemma (which is proved by adapting the scalar proof of [N, p. 139]).

KRONECKER'S LEMMA: *Let X be a Banach space. If $\{b_n\}$ is a sequence of positive numbers increasing to ∞, and $\{x_n\}$ is a sequence in X such that $\sum_{k=1}^{\infty}(x_k/b_k)$ converges strongly (is bounded), then $\|(1/b_n)\sum_{k=1}^{n}x_k\|$ converges to 0 (respectively, is bounded).*

THEOREM 2.14: *Let T be a contraction in a Banach space X, and $0 < \alpha < 1$. If $y \in (I-T)^\alpha\overline{(I-T)X}$, then $\lim_n \|(1/n^{1-\alpha})\sum_{k=1}^{n}T^k y\| = 0$.*

Proof: Put $b_n = 1/(n+1)a_{n+1}^{(1-\alpha)}$. By Lemma 2.5, b_n increases to ∞. Theorem 2.7 and Kronecker's lemma yield $\lim_n \|(1/b_n)\sum_{k=1}^{n}T^k y\| = 0$. Replacing α by $1-\alpha$ in (2.2), we obtain $\lim_n n^{1-\alpha}/b_n = C > 0$, which yields the theorem. ∎

COROLLARY 2.15: *Let T be a mean ergodic contraction in a Banach space X, and $0 < \alpha < 1$. If $y \in (I-T)^\alpha X$, then $\lim_n \|(1/n^{1-\alpha})\sum_{k=1}^n T^k y\| = 0$.*

Remarks: 1. The converse of Corollary 2.15 is false — counterexamples, in a Hilbert space, will be given in section 3 (for a unitary operator) and in section 4 (for a symmetric contraction).

2. Without mean ergodicity, Corollary 2.15 may fail: Let $X = \ell_1$, and denote the standard basis by $e_0, e_1, \dots, e_j, \dots$. Let T be the shift $T(\sum_{j=0}^\infty t_j e_j) = \sum_{j=0}^\infty t_j e_{j+1}$, and take $y = (I-T)^\alpha e_0 = e_0 - \sum_{j=1}^\infty a_j^{(\alpha)} e_j$. Then

$$\sum_{k=0}^n T^k y = e_0 + \sum_{k=1}^n \Big(1 - \sum_{j=1}^k a_j^{(\alpha)}\Big) e_k - \sum_{k=n+1}^\infty \Big(\sum_{j=k-n}^k a_j^{(\alpha)}\Big) e_k.$$

Since all terms are on different basis vectors, Lemma 2.5 and (2.2) yield

$$\Big\|\sum_{k=0}^n T^k y\Big\|_1 \ge \sum_{k=1}^n \Big(\sum_{j=k+1}^\infty a_j^{(\alpha)}\Big) = \frac{1}{\alpha}\sum_{k=1}^n (k+1) a_{k+1}^{(\alpha)} \ge C \sum_{k=1}^n k^{-\alpha} \ge C' n^{1-\alpha}.$$

PROPOSITION 2.16: *Let $0 < \alpha < 1$. Then there exists a constant C_α such that for every contraction T on a Banach space we have*

$$\sup_n \Big\| \frac{1}{n^{1-\alpha}} \sum_{k=1}^n T^k (I-T)^\alpha \Big\| \le C_\alpha.$$

Hence

$$\sup_n \Big\| \frac{1}{n^{1-\alpha}} \sum_{k=1}^n T^k y \Big\| < \infty \quad \text{for } y \in (I-T)^\alpha X.$$

Proof: Let b_n be as in Theorem 2.14, and apply Kronecker's lemma to Proposition 2.4(ii) (the constant does not depend on T by the proof of Kronecker's lemma). Then use (2.2). ∎

THEOREM 2.17: *Let T be a contraction in a Banach space X, and let $0 < \beta < 1$. If $\sup_n \|(1/n^{1-\beta})\sum_{k=1}^n T^k y\| < \infty$, then $y \in (I-T)^\alpha X$ for every $0 < \alpha < \beta$.*

Proof: Fix $0 < \alpha < \beta$, and let $\gamma = 1 - \alpha$. We use Abel's summation by parts to prove convergence of $\sum_{k=1}^n T^k y / k^\gamma$. Denote $y_0 = 0$ and $y_k = \sum_{j=1}^k T^j y$. For $j < n-1$ we have

$$\begin{aligned}\sum_{k=j+1}^n \frac{T^k y}{k^\gamma} &= \sum_{k=j+1}^n \frac{y_k - y_{k-1}}{k^\gamma} = \frac{y_n}{n^\gamma} - \frac{y_j}{(j+1)^\gamma} + \sum_{k=j+1}^{n-1} \Big(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\Big) y_k \\ &= \frac{n^{1-\beta}}{n^\gamma}\frac{y_n}{n^{1-\beta}} - \frac{j^{1-\beta}}{(j+1)^\gamma}\frac{y_j}{j^{1-\beta}} + \sum_{k=j+1}^{n-1}\Big(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\Big) k^{1-\beta}\frac{y_k}{k^{1-\beta}}.\end{aligned}$$

By assumption, $\sup_n \|y_n/n^{1-\beta}\| = C < \infty$, so the first two terms tend to 0 as $j, n \to \infty$ (since $\gamma > 1-\beta$). For $j < n-1$ we have

$$\Big\|\sum_{k=j+1}^{n-1}\Big(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\Big)k^{1-\beta}\frac{y_k}{k^{1-\beta}}\Big\| \le$$
$$C\sum_{k=j+1}^{n-1}\Big(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\Big)k^{1-\beta} \le \gamma C \sum_{k=j+1}^{\infty}\frac{1}{k^{\gamma+\beta}},$$

which tends to 0 as $j \to \infty$ (since $\gamma + \beta > 1$). Hence $\sum_{k=1}^{n} T^k y/k^\gamma$ converges. By (2.6) (which does not require the ergodicity assumption of Theorem 2.11), $\sum_{j=1}^{\infty}(j+1)a_{j+1}^{(\gamma)}T^j y$ converges, which yields that $y \in (I-T)^\alpha X$ by Proposition 2.10. ∎

Remark: Theorem 2.17 shows that if $(I-T)X$ is not closed, then for any $0 < \alpha < 1$ there is $y \in \overline{(I-T)X}$ with $\sup_n \|(1/n^{1-\alpha})\sum_{k=1}^n T^k y\| = \infty$: Let $0 < \delta < \alpha$, and take $y \in \overline{(I-T)X}$ which is not in $(I-T)^\delta X$ (exists by Proposition 2.1). Theorem 2.23(iii) below improves this result, by yielding a y not dependent on α.

PROPOSITION 2.18: *Let T be a contraction in a reflexive Banach space X, $y \in X$, and $0 < \alpha < 1$. Then*

$$\frac{1}{n^{1-\alpha}}\sum_{k=1}^{n} T^k y \overset{weakly}{\to} 0 \quad \textit{if and only if} \quad \sup_n \Big\|\frac{1}{n^{1-\alpha}}\sum_{k=1}^{n} T^k y\Big\| < \infty.$$

Proof: The weak convergence clearly implies the boundedness condition.

Assume $\sup_n \|(1/n^{1-\alpha})\sum_{k=1}^n T^k y\| < \infty$. Then $\|(1/n)\sum_{k=1}^n T^k y\| \to 0$, which yields $y \in \overline{(I-T)X}$. By reflexivity, $\Big\{(1/n^{1-\alpha})\sum_{k=1}^n T^k y\Big\}_{n>0}$ is weakly sequentially compact. Let the subsequence $(1/n_j^{1-\alpha})\sum_{k=1}^{n_j} T^k y$ converge weakly, say to x. Since a closed subspace is weakly closed, also $x \in \overline{(I-T)X}$. Now

$$(I-T)x = \text{w-}\lim_j \frac{1}{n_j^{1-\alpha}}\sum_{k=1}^{n_j}(I-T)T^k y = \text{w-}\lim_j \frac{1}{n_j^{1-\alpha}}(Ty - T^{n_j+1}y) = 0.$$

Hence $Tx = x$, so $x = 0$. This yields the required weak convergence to 0. ∎

Remarks: 1. Strong convergence need not hold in the proposition, and in Theorem 2.17 y need not be in $(I-T)^\beta X$: Let $\{e_j\}$ be the standard orthonormal basis of ℓ_2, and T the shift. Then $\|\sum_{k=1}^n T^k e_1\| = \sqrt{n}$, so for $\alpha = \frac{1}{2}$ the boundedness condition of the proposition is satisfied without strong convergence. Corollary

2.15 shows that e_1 cannot be in $(I-T)^{\frac{1}{2}}\ell_2$, though the assumptions of Theorem 2.17 are satisfied for $\beta = \frac{1}{2}$, and $e_1 \in (I-T)^\alpha \ell_2$ for $0 < \alpha < \frac{1}{2}$.

2. The shift in ℓ_2 of the previous remark provides an example where $(I-T)X \neq \bigcap_{0<\alpha<1}(I-T)^\alpha X$. Let $z = (I-T)^{\frac{1}{2}}e_1$. By the above, $z \in (I-T)^\beta \ell_2$ for $\frac{1}{2} < \beta < 1$, so by Proposition 2.1(i), $z \in \bigcap_{0<\alpha<1}(I-T)^\alpha \ell_2$. If $z = (I-T)y$, then $(I-T)^{1/2}[e_1 - (I-T)^{1/2}y] = 0$. Since $\ker(I-T) = \{0\}$, Proposition 2.1(iv) yields $e_1 = (I-T)^{\frac{1}{2}}y$ — a contradiction, so $z \notin (I-T)\ell_2$.

Lemma 2.19: *Let $\{x_j\}$ be a sequence in a Banach space, and $\{b_j\}$ a sequence of positive numbers monotonely decreasing to 0. If $\{\sum_{j=1}^n x_j\}$ is bounded, then $\sum_{j=1}^\infty b_j x_j$ converges strongly.*

Proof: Let $y_n = \sum_{j=1}^n x_j$. By assumption $\|y_n\| \leq C$ for every n. For $n > k$ we have

$$\begin{aligned}\left\|\sum_{j=k+1}^{n} b_j x_j\right\| &= \left\|\sum_{j=k+1}^{n} b_j(y_j - y_{j-1})\right\| \\ &\leq \left\|b_n y_n - b_{k+1}y_k + \sum_{j=k+1}^{n-1}(b_j - b_{j+1})y_j\right\| \leq 2b_{k+1}C,\end{aligned}$$

which tends to 0 as $k \to \infty$. Hence $\{\sum_{j=1}^n b_j x_j\}$ is Cauchy. ∎

Proposition 2.20: *Let T be a contraction in a Banach space X, and $0 < \alpha < 1$. If $y \in (I-T)^\alpha X$, then the series $\sum_{j=1}^\infty T^j y/j$ converges strongly.*

Proof: Proposition 2.4(ii) with (2.6) yield the boundedness of $\{\sum_{j=1}^n T^j y/j^{1-\alpha}\}_n$, and we apply Lemma 2.19 with $b_j = j^{-\alpha}$. ∎

Proposition 2.21: *Let T be a contraction in a Banach space X, and let $y \in X$. If $\sum_{j=1}^\infty T^j y/j$ converges strongly, then*

$$\lim_{\alpha\to 0^+}\left\|\frac{1}{\alpha}[(I-T)^\alpha - I]y + \sum_{j=1}^{\infty}\frac{T^j y}{j}\right\| = 0.$$

Proof: Put $A_\alpha y := \frac{1}{\alpha}[(I-T)^\alpha - I]y$. Using (2.1), we obtain

$$\begin{aligned}A_\alpha y &= -\frac{1}{\alpha}\sum_{j=1}^{\infty} a_j^{(\alpha)} T^j y = -Ty - \sum_{j=2}^{\infty}\frac{\Pi_{k=1}^{j-1}(k-\alpha)}{j!}T^j y \\ &= -Ty - \sum_{j=2}^{\infty}\left[\Pi_{k=1}^{j-1}\left(1 - \frac{\alpha}{k}\right)\right]\frac{T^j y}{j}.\end{aligned}$$

For $j \geq 2$ and $0 < \alpha < 1$ define $c_{j,\alpha} := 1 - \Pi_{k=1}^{j-1}(1 - \alpha/k)$. Then $0 < c_{j,\alpha} < 1$. For α fixed $c_{j,\alpha}$ is increasing in j, and for j fixed $c_{j,\alpha}$ decreases to 0 as α decreases to 0^+. Hence, for every integer $N > 0$ we have

$$\limsup_{\alpha\to 0^+}\left\|\frac{1}{\alpha}[(I-T)^\alpha - I]y + \sum_{j=1}^{\infty}\frac{T^j y}{j}\right\| = \limsup_{\alpha\to 0^+}\left\|\sum_{j=2}^{\infty} c_{j,\alpha}\frac{T^j y}{j}\right\|$$
$$\leq \limsup_{\alpha\to 0^+}\left\|\sum_{j=N+1}^{\infty} c_{j,\alpha}\frac{T^j y}{j}\right\|.$$

For $j \geq n$ denote $x_{n,j} := \sum_{k=n}^{j} T^k y/k$. By the given convergence, for $\varepsilon > 0$ there is N such that $\|x_{n,j}\| < \varepsilon$ for $j \geq n \geq N$. For any $0 < \alpha < 1$ and $K > N$, the monotonicity of $c_{j,\alpha}$ in j yields

$$\left\|\sum_{j=N+1}^{K} c_{j,\alpha}\frac{T^j y}{j}\right\| = \left\|\sum_{j=N+1}^{K} c_{j,\alpha}(x_{N,j} - x_{N,j-1})\right\|$$
$$\leq \|c_{K,\alpha}x_{N,K}\| + \|c_{N+1,\alpha}x_{N,N}\| + \left\|\sum_{j=N+1}^{K-1}(c_{j,\alpha} - c_{j+1,\alpha})x_{N,j}\right\|$$
$$\leq 3\varepsilon.$$

Hence

$$\limsup_{\alpha\to 0^+}\left\|\sum_{j=N+1}^{\infty} c_{j,\alpha}\frac{T^j y}{j}\right\| \leq 3\varepsilon,$$

which proves the proposition. ∎

THEOREM 2.22: *Let T be a contraction in a Banach space X, and denote $Y = \overline{(I-T)X}$. Then*

(i) *$\{(I-T)^r_{|Y} : r \geq 0\}$ is a strongly continuous (at 0) semigroup on Y (with $(I-T)^0_{|Y} = I_{|Y}$).*

(ii) *If $\sum_{j=1}^{\infty} T^j y/j$ converges strongly, then y is in the domain of the infinitesimal generator A, and $Ay = -\sum_{j=1}^{\infty} T^j y/j$.*

(iii) *$(I-T)^\alpha X$ is contained in the domain of A, for every $0 < \alpha < 1$.*

Proof: (i) Fix some $0 < \beta < 1$. By Proposition 2.1(ii), $(I-T)^\beta X$ is dense in Y, so Proposition 2.20 yields the convergence of $\sum_{j=1}^{\infty} T^j y/j$ for y in a dense subset of Y, and for those y, Proposition 2.21 yields $\lim_{\alpha\to 0^+}(I-T)^\alpha y = y$. Since $\|(I-T)^\alpha\| \leq 2$ for $\alpha < 1$, we obtain the strong continuity at 0.

(ii) is Proposition 2.21, and (iii) follows from (ii) and Proposition 2.20. ∎

THEOREM 2.23: *Let T be a contraction in a Banach space X. If $(I-T)X$ is not closed, then:*

(i) *There exists $y \in \overline{(I-T)X}$ such that $\sum_{j=1}^{n} T^j y/j$ does not converge as $n \to \infty$.*

(ii) $\bigcup_{0<\alpha<1} (I-T)^\alpha X \neq \overline{(I-T)X}$.

(iii) *There exists $y \in \overline{(I-T)X}$ such that $\sup_n \|(1/n^{1-\alpha}) \sum_{k=1}^{n} T^k y\| = \infty$ for every $0 < \alpha < 1$.*

Proof: (i) Denote $Y = \overline{(I-T)X}$, and let $S = T_{|Y}$. Suppose that $\sum_{j=1}^{\infty} T^j y/j$ converges strongly for every $y \in Y$, say to Hy. By the Banach–Steinhaus theorem, H is a bounded operator on Y. The previous theorem yields that H is the infinitesimal generator of the semigroup $\{(I-S)^r : r \geq 0\}$. We then have a strongly continuous semigroup on Y with a bounded generator, so by [DuS, Corollary VIII.1.9] it is uniformly continuous, i.e., $\|(I-S)^\alpha - I\| \underset{\alpha \to 0^+}{\longrightarrow} 0$. Hence, for $0 < \alpha < 1$ small enough, $(I-S)^\alpha$ is invertible on Y, so $Y = (I-S)^\alpha Y$. By Proposition 2.1(iii), $(I-S)Y = Y$, which yields $Y \subset (I-T)X$. Hence $(I-T)X = Y$, contradicting the assumption that $(I-T)X$ is not closed.

(ii) follows from (i) and Proposition 2.20. (iii) follows from (ii) and Theorem 2.17. ∎

Remark: For T induced by a probability preserving transformation, part (i) of the theorem was proved by Halmos [H]. For a unitary operator U in a Hilbert space H, Campbell [C] proved the norm convergence of the **two-sided Hilbert transform** $\lim_{n\to\infty} \sum_{0<|k|\leq n} U^k y/k$, for every $y \in H$.

3. Pointwise convergence for fractional coboundaries

In this section we look at the pointwise convergence, instead of the norm convergence treated in the previous section, when T is a Dunford–Schwartz (DS) operator on L_1 of a probability space (i.e., T is a contraction of L_1 such that $\|Tf\|_\infty \leq \|f\|_\infty$ for every $f \in L_\infty$). If θ is a measure preserving transformation in a probability space (Ω, Σ, μ), then the operator $Tf = f \circ \theta$ is a DS operator on $L_1(\mu)$. More generally, any Markov operator P with an invariant probability yields a positive DS operator. We denote by $\mathbf{T}$ the linear modulus of a linear contraction T on L_1 [Kr, p. 159]. It is known that if T is a DS operator, so is $\mathbf{T}$. The Dunford–Schwartz theorem says that for a DS operator in a probability space, the pointwise ergodic theorem holds, i.e., for every $f \in L_1$, the averages $\frac{1}{n} \sum_{k=1}^{n} T^k f$ converge a.e. This implies the mean ergodicity of T. Now the same applies to $\mathbf{T}$ (note that the Dunford–Schwartz proof is based on proving the maximal inequality for $\mathbf{T}$).

THEOREM 3.1: *Let T be a contraction of $L_1(\mu)$ with mean ergodic modulus $\mathbf{T}$. If $f \in (I-T)^\alpha L_1$ for some $0 < \alpha < 1$, then the one-sided Hilbert transform $\sum_{k=1}^{\infty} T^k f/k$ converges a.e.*

Proof: Note that norm convergence follows from Proposition 2.20.

We use Abel's summation by parts. Let $S_0 f = 0$, and $S_k f = \sum_{j=1}^{k} T^j f$ for $k > 0$. For $0 < \gamma \le 1$ we have

$$\sum_{k=1}^{n} \frac{T^k f}{k^\gamma} = \sum_{k=1}^{n} \frac{S_k f - S_{k-1} f}{k^\gamma} = \frac{S_n f}{n^\gamma} + \sum_{k=1}^{n-1} \Big(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\Big) S_k f. \tag{3.1}$$

We want to prove a.e. convergence for $\gamma = 1$. Since T has mean ergodic modulus, $S_n f/n$ converges a.e. and in norm [ÇLi] (so the limit is 0, because $f \in \overline{(I-T)L_1}$). For the series, we write

$$\sum_{k=1}^{n-1} \Big|\Big(\frac{1}{k^\gamma} - \frac{1}{(k+1)^\gamma}\Big) S_k f\Big| \le \gamma \sum_{k=1}^{n-1} \frac{1}{k^{1+\gamma}} |S_k f| \le \gamma \sum_{k=1}^{n-1} \frac{1}{k^{\alpha+\gamma}} \Big|\frac{S_k f}{k^{1-\alpha}}\Big|.$$

Since T is mean ergodic, and $f \in (I-T)^\alpha L_1$, Corollary 2.15 yields $\lim_k \|S_k f/k^{1-\alpha}\|_1 = 0$. By Lebesgue's theorem, for $\gamma > 1-\alpha$ we have

$$\int \sum_{k=1}^{\infty} \frac{1}{k^{\alpha+\gamma}} \Big|\frac{S_k f}{k^{1-\alpha}}\Big| d\mu = \sum_{k=1}^{\infty} \frac{1}{k^{\alpha+\gamma}} \Big\|\frac{S_k f}{k^{1-\alpha}}\Big\|_1 < \infty.$$

Hence

$$\sum_{k=1}^{\infty} \frac{1}{k^{\alpha+\gamma}} \Big|\frac{S_k f}{k^{1-\alpha}}\Big| < \infty \quad \text{a.e. for } \gamma > 1-\alpha,$$

so the series on the right hand side of (3.1) is a.e. absolutely convergent for $\gamma = 1$. ∎

Remarks: 1. The series on the right hand side of (3.1) converges a.e. also for $1-\alpha < \gamma < 1$, so the a.e. convergence of $\sum_{k=1}^{n} T^k f/k^\gamma$ depends on the a.e. convergence of $S_n f/n^\gamma$. If $\{g_n\}$ are i.i.d. with $g_0 \in L_1$ but not in any L_p with $p > 1$, and θ is the shift, i.e., $g_n = g \circ \theta^n$, the Borel–Cantelli lemma shows that for $\gamma < 1$ we have a.e. divergence of $g \circ \theta^n/n^\gamma$. For T induced by θ, we take $f = (I-T)g \in (I-T)^\alpha L_1$, and see that when $\gamma < 1$ a.e. convergence in (3.1) may fail.

2. It is known [P, p. 94] (see also [DRo]) that for every θ ergodic measure preserving in a non-atomic probability space there is $f \in L_\infty$ with $\int f d\mu = 0$, such that $\sum_{k=1}^{n} f \circ \theta^k/k$ is a.e. divergent; this is in contrast with the a.e. convergence of the two-sided ergodic Hilbert transform for every $f \in L_1$, for any *invertible* ergodic probability preserving transformation (see [P]).

3. For T induced by an ergodic measure preserving transformation, Assani [A, Theorem 9] proved a.e. convergence of the rotated one-sided Hilbert transform for functions which he calls **Wiener–Wintner functions** (defined by an appropriate L_1-norm inequality).

4. If C and D are the conservative and dissipative parts of the modulus $\mathbf{T}$ of a contraction T on L_1, then $\sum_{k=1}^{\infty} |T^k f|/k^\gamma \leq \sum_{k=1}^{\infty} \mathbf{T}^k |f| < \infty$ a.e. on D, for every $f \in L_1$ and $0 \leq \gamma \leq 1$. The previous result is therefore of interest only on C, i.e., when $\mathbf{T}$ is not dissipative. For conditions equivalent to mean ergodicity of $\mathbf{T}$, see [Kr, p. 175]. Note that D need not be absorbing, so there may be a "contribution of mass" from D to C.

Let T be a Dunford–Schwartz operator. The conservative and dissipative parts of its modulus $\mathbf{T}$ are both absorbing [Kr, p. 131], so the previous remark shows that we have to deal only with the case that $\mathbf{T}$ is conservative, i.e., $\mathbf{T}1 = 1$ a.e. (since the measure is finite).

The modulus $\mathbf{T}$ of a DS contraction is also a contraction in each of the $L_p(\mu)$ spaces, $1 < p < \infty$ (for a simple proof see [Kr, p. 65]), and hence so is T. For $g \in L_p$ we have $\mathbf{T}^n(|g|^p)/n \to 0$ a.e., by the ergodic theorem. By [Kr, p. 65]

$$|T^n g|/n^{1/p} \leq \mathbf{T}^n(|g|)/n^{1/p} \leq [\mathbf{T}^n(|g|^p)/n]^{1/p} \to 0.$$

Hence, for $f \in (I-T)L_p$ (i.e., with $\sup_n \| \sum_{k=0}^{n} T^k f \| < \infty$) we have $S_n f/n^{1/p} \to 0$ a.e. — an improvement upon the information given by the ergodic theorem. For a given DS operator T, we want to obtain a larger class of functions of L_p, which contains $(I-T)L_p$, with this better convergence (when $\mathbf{T}$ is conservative). Note that we cannot have a smaller power of n which will give the improved convergence for every $f \in (I-T)L_p$ and every T (induced by a probability preserving transformation).

Below, we use the assumption $f \in (I-T)^\alpha L_p$ for $1 < p < \infty$ and $0 < \alpha < 1$, which, by (2.6) and Corollary 2.12, is equivalent to

$$\liminf_{N\to\infty} \Big\| \sum_{j=1}^{N} \frac{T^j f}{j^{1-\alpha}} \Big\|_p < \infty.$$

Thus, we will obtain a.e. convergence results from norm assumptions.

THEOREM 3.2: *Let T be a Dunford–Schwartz operator in $L_1(\mu)$ of a probability space, and fix $1 < p < \infty$, with dual index $q = p/(p-1)$. Let $0 < \alpha < 1$, and $f \in (I-T)^\alpha L_p$.*

(i) *If $\alpha > 1 - 1/p = 1/q$, then $(1/n^{1/p}) \sum_{k=0}^{n-1} T^k f \to 0$ a.e.*

(ii) *If $\alpha = 1 - 1/p = 1/q$, then $(1/n^{1/p}(\log n)^{1/q}) \sum_{k=0}^{n-1} T^k f \to 0$ a.e.*

(iii) *If $\alpha < 1 - 1/p = 1/q$, then $(1/n^{1-\alpha}) \sum_{k=0}^{n-1} T^k f \to 0$ a.e.*

Proof: Denote $a_j^{(\alpha)}$ by a_j. We know that $f = g - \sum_{j=1}^\infty a_j T^j g$ with $g \in L_p$, and the series is a.e. absolutely convergent (by Lebesgue's theorem). We may take $g \in \overline{(I-T)L_p}$, so we have $\frac{1}{n}\sum_{k=1}^n T^k g \to 0$ a.e. By the absolute convergence, we can reorder the terms in $\sum_{k=0}^{n-1} T^k f$ to represent it as $A_n + B_n + C_n$, with

$$A_n := g + \sum_{j=1}^{n-1} [1 - (a_1 + \ldots + a_j)] T^j g = g + \sum_{j=1}^{n-1} \Big(\sum_{k=j+1}^\infty a_k \Big) T^j g, \tag{3.2}$$

$$B_n := -\sum_{j=1}^{n} T^{n-1}(a_j + a_{j+1} + \ldots + a_n) T^j g = -\sum_{j=1}^{n} T^{n-1} \Big(\sum_{k=j}^{n} a_k \Big) T^j g, \tag{3.3}$$

$$C_n := -\sum_{j=n+1}^\infty a_j (T^j g + \ldots + T^{j+n-1} g) = -\sum_{j=n+1}^\infty a_j T^j \Big(\sum_{k=0}^{n-1} T^k g \Big). \tag{3.4}$$

We will prove separately, in a succession of steps:

(I) $A_n/n^{1-\alpha} \to 0$ a.e. in both cases (i) and (ii).

(II) $C_n/n^{1/p} \to 0$ a.e. in cases (i) and (ii).

(III) $B_n/n^{1/p} \to 0$ a.e. in case (i), and $B_n/[n^{1/p}(\log n)^{1/q}] \to 0$ a.e. in case (ii).

(IV) $A_n/n^{1-\alpha} \to 0$ a.e. and $C_n/n^{1-\alpha} \to 0$ a.e. in case (iii).

(V) $B_n/n^{1-\alpha} \to 0$ a.e. in case (iii).

STEP I: We prove that $A_n/n^{1-\alpha} \to 0$ a.e. in both cases (i) and (ii), using the following lemmas. The proof of the first one is standard.

LEMMA 3.3: *Let $\{d_j\}$ be a sequence of real numbers with $\sup_n \frac{1}{n}\sum_{j=1}^n |d_j| < \infty$ and $\frac{1}{n}\sum_{j=1}^n d_j \to 0$. If $c_j \to c$, then also $\frac{1}{n}\sum_{j=1}^n c_j d_j \to 0$.*

LEMMA 3.4: *If $(1/n)\sum_{j=1}^n d_j \to 0$, then for any $0 < \alpha < 1$ we have that $(1/n^{1-\alpha}) \sum_{j=1}^n d_j/j^\alpha \to 0$.*

Proof: Let $s_0 = 0$, and $s_j = \sum_{k=1}^{j} d_k$. Then

$$\left|\frac{1}{n^{1-\alpha}}\sum_{j=1}^{n}\frac{d_j}{j^\alpha}\right| \le \frac{1}{n^{1-\alpha}}\sum_{j=1}^{n-1}\left|s_j\left(\frac{1}{j^\alpha}-\frac{1}{(j+1)^\alpha}\right)\right| + \frac{|s_n|}{n}$$
$$\le \frac{\alpha}{n^{1-\alpha}}\sum_{j=1}^{n-1}|s_j|\frac{1}{j^{1+\alpha}} + \frac{|s_n|}{n}.$$

Let $\epsilon > 0$. There is k such that $|s_j| < \epsilon j$ for $j > k$. Using the estimate $\sum_{j=2}^{n-1} j^{-\alpha} \le n^{1-\alpha}/(1-\alpha)$, we show that

$$\limsup_n \left|\frac{1}{n^{1-\alpha}}\sum_{j=1}^{n}\frac{d_j}{j^\alpha}\right| \le \epsilon\alpha/(1-\alpha).$$

Continuation of the proof of Step I: By (2.2), $j^{1+\alpha}a_j$ converges. By the ergodic theorem and Lemma 3.3, we obtain that $\frac{1}{n}\sum_{j=1}^{n}(j+1)^{1+\alpha}a_{j+1}T^j g \to 0$ a.e. By inserting $\sum_{k=j}^{\infty} a_k = \frac{1}{\alpha} j a_j$ (from Lemma 2.5) into (3.2) and using Lemma 3.4, we obtain (with no aditional restriction on $\alpha < 1$)

$$\frac{1}{n^{1-\alpha}}A_n = \frac{1}{n^{1-\alpha}}g + \frac{1}{\alpha n^{1-\alpha}}\sum_{j=1}^{n}(j+1)a_{j+1}T^j g \to 0 \quad \text{a.e.}$$

STEP II: We prove that $C_n/n^{1/p} \to 0$ a.e. in case (ii). This yields the result also for case (i), since $(I-T)^\alpha L_p \subset (I-T)^{1/q}L_p$ for $\alpha > 1/q$. We need the following lemma.

LEMMA 3.5: *Let $\{d_j\}$ be a sequence of non-negative numbers, and $0 < \alpha < 1$. If $\sup_n (1/n)\sum_{j=1}^{n} d_j = M < \infty$, then $\sup_n n^\alpha \sum_{j=n+1}^{\infty} d_j/j^{1+\alpha} \le (1+\alpha)M$.*

Proof: Again, let $s_0 = 0$, and $s_j = \sum_{i=1}^{j} d_i$. Summation by parts yields

$$\sum_{j=n+1}^{N}\frac{d_j}{j^{1+\alpha}} = \sum_{j=n+1}^{N}\frac{s_j - s_{j-1}}{j^{1+\alpha}}$$
$$= \sum_{j=n+1}^{N-1} s_j\left(\frac{1}{j^{1+\alpha}} - \frac{1}{(j+1)^{1+\alpha}}\right) + \frac{s_N}{N^{1+\alpha}} - \frac{s_n}{(n+1)^{1+\alpha}}.$$

The assumpion $s_n \le nM$ and the positivity of s_n yield

$$\sum_{j=n+1}^{\infty}\frac{d_j}{j^{1+\alpha}} \le M(1+\alpha)\sum_{j=n+1}^{\infty}\frac{1}{j^{1+\alpha}} \le \frac{M(1+\alpha)}{\alpha n^\alpha},$$

which proves the lemma.

Continuation of the proof of Step II: By the maximal inequality for $\mathbf{T}$, the function $g^* = \sup_n \frac{1}{n}\sum_{k=0}^{n-1}\mathbf{T}^k(|g|)$ is in $L_p \subset L_1$. Substituting in (3.4) we get $|C_n| \le n\sum_{j=n+1}^{\infty} a_j \mathbf{T}^j g^*$, with absolute convergence of the series a.e. Using $\alpha = 1/q$ (case (ii)) and $a_n \sim 1/n^{1+\alpha}$ we obtain

$$\frac{|C_n|}{n^{1/p}} \le n^{1/q}\sum_{j=n+1}^{\infty} a_j\mathbf{T}^j g^* = n^{\alpha}\sum_{j=n+1}^{\infty} a_j\mathbf{T}^j g^* \le Kn^{\alpha}\sum_{j=n+1}^{\infty}\frac{1}{j^{1+\alpha}}\mathbf{T}^j g^*.$$

The function $g^{**} := \sup_n(1/n)\sum_{k=0}^{n-1}\mathbf{T}^k(g^*)$ is in L_p since g^* is, so by the previous lemma we have $|C_n|/n^{1/p} \le K'g^{**}$ a.e.

The operators $C_n(g)$, defined on L_p by (3.4), satisfy $\sup_n |C_n(g)|/n^{1/p} \le K'g^{**}$ a.e. By (2.2) and Lemma 2.5, $C_n(g)/n^{1/p}$ converges for g constant. By the Banach principle, it is enough to prove that $C_n(g)/n^{1/p} \to 0$ a.e. for $g = h - Th$ with $h \in L_p$. For such a g, we easily compute that

$$-C_n(g) = a_{n+1}T^{n+1}\sum_{k=0}^{n-1}T^k h + \sum_{k=0}^{n-1}T^k\Big[\sum_{j=n+2}^{\infty}(a_j - a_{j-1})T^j h\Big].$$

By (2.1),

$$a_{j-1} - a_j = \frac{1+\alpha}{j}a_{j-1} \quad \text{for } j \ge 3.$$

Hence

$$\frac{|C_n(g)|}{n^{1/p}} \le n^{1-1/p}a_{n+1}\mathbf{T}^{n+1}h^* + \frac{1+\alpha}{n^{1/p}}n\sum_{j=n+2}^{\infty}\frac{a_{j-1}}{j}\mathbf{T}^j h^*. \tag{3.5}$$

Since $\alpha = 1/q$, (2.2) yields for the first term on the right-hand side of (3.5) a majorization by $K_1\mathbf{T}^{n+1}h^*/n$, which converges to 0 a.e. (by the ergodic theorem). Using (2.2), and Lemma 3.5 for the second inequality below, we obtain a majorization of the second term of the right hand side of (3.5) by

$$K_2 n^{1/q}\sum_{j=n+2}^{\infty}\frac{1}{j^{2+1/q}}\mathbf{T}^j h^* \le K_2\frac{n^{1/q}}{n}\sum_{j=n+2}^{\infty}\frac{1}{j^{1+1/q}}\mathbf{T}^j h^* \le Mh^{**}K_2\frac{n^{1/q}}{n} \to 0.$$

STEP III: We now deal with B_n. Using again $\sum_{k=j}^{\infty} a_k = (1/\alpha)ja_j$, (3.3) yields

$$\begin{aligned} B_n &= -T^{n-1}\sum_{j=1}^{n}\frac{1}{\alpha}(ja_j - (n+1)a_{n+1})T^j g \\ &= -T^{n-1}\sum_{j=1}^{n}\frac{1}{\alpha}ja_jT^j g + \frac{1}{\alpha}(n+1)a_{n+1}\sum_{k=n}^{2n-1}T^k g. \end{aligned} \tag{3.6}$$

Since $g \in \overline{(I-T)L_p}$, the averages $(1/n)\sum_{k=0}^{n-1} T^k g$ tend to 0 a.e. The asymptotic behaviour $a_n \sim 1/n^{1+\alpha}$ yields for the last term on the right hand side of (3.6) that

$$\frac{1}{n^{1-\alpha}}(n+1)a_{n+1}\sum_{k=n}^{2n-1} T^k g = \frac{1}{n^{1-\alpha}}(n+1)a_{n+1}n\left[\frac{1}{n}\sum_{k=0}^{2n-1} T^k g - \frac{1}{n}\sum_{k=0}^{n-1} T^k g\right] \to 0 \text{ a.e.}$$

In order to deal with the first term of the right hand side of (3.6), we need the following claim, which implies the theorem.

Claim 1: (i) *If $\alpha > 1/q$, then*

$$\text{(3.7)} \qquad \textit{for every } h \in L_p, \quad \frac{1}{n^{1/p}} T^{n-1}\sum_{k=1}^{n} k a_k T^k h \to 0 \quad \text{a.e.}$$

(ii) *If $\alpha = 1/q$, then*

$$\text{(3.8)} \qquad \textit{for every } h \in L_p, \quad \frac{1}{n^{1/p}(\log n)^{1/q}} T^{n-1}\sum_{k=1}^{n} k a_k T^k h \to 0 \quad \text{a.e.}$$

Proof of Claim 1: For $h \in L_p$ let $\tilde{h} = \sup_n 1/n\sum_{k=0}^{n-1}\mathbf{T}^k(|h|^p)$, which is a.e. finite. Then $\frac{1}{n}\sum_{j=n}^{2n-1}\mathbf{T}^j(|h|^p) \le 3\tilde{h}$. Using again $a_k \sim k^{-(1+\alpha)}$, we have $\sum_{k=1}^{n} k^q a_k^q \le K\sum_{k=1}^{n} 1/k^{\alpha q}$. By Hölder's inequality

$$\begin{aligned}
\left|T^{n-1}\sum_{k=1}^{n} k a_k T^k h\right| &= \left|\sum_{j=n}^{2n-1}(j+1-n)a_{j+1-n}T^j h\right| \\
&\le \left[\sum_{k=1}^{n} k^q a_k^q\right]^{1/q}\left[\sum_{j=n}^{2n-1}|T^j h|^p\right]^{1/p} \\
&\le \left[\sum_{k=1}^{n} k^q a_k^q\right]^{1/q}\left[\sum_{j=n}^{2n-1}\mathbf{T}^j(|h|^p)\right]^{1/p} \\
&\le n^{1/p}\left[K\sum_{k=1}^{n}\frac{1}{k^{\alpha q}}\right]^{1/q}(3\tilde{h})^{1/p}.
\end{aligned}$$

In case (i), $\alpha q > 1$, so the last series converges, and we obtain

$$\text{for every } h \in L_p, \quad \sup_n\left|\frac{1}{n^{1/p}}T^{n-1}\sum_{k=1}^{n} k a_k T^k h\right| < \infty \quad \text{a.e.}$$

In case (ii), the estimate $\sum_{k=1}^{n} 1/k \le 1 + \log n$ yields

$$\text{for every } h \in L_p, \quad \sup_n \left| \frac{1}{n^{1/p}(\log n)^{1/q}} T^{n-1} \sum_{k=1}^{n} k a_k T^k h \right| < \infty \quad \text{a.e.}$$

By the Banach principle, it is now enough to prove (3.7) or (3.8), as the case is, only for h in a dense subspace of L_p. For h invariant, $a_k \sim 1/k^{1+\alpha}$ yields $\sum_{k=1}^{n} k a_k \le K \sum_{k=1}^{n} 1/k^{\alpha} \sim n^{1-\alpha}$, which implies the desired convergence to 0. We now take $h \in (I - T)L_p$, i.e., $h = \hat{h} - T\hat{h}$, and obtain

$$\tag{3.9} \frac{1}{n^{1/p}} T^{n-1} \sum_{k=1}^{n} k a_k T^k h =$$

$$\frac{1}{n^{1/p}} T^{n-1} \sum_{k=2}^{n} [k a_k - (k-1) a_{k-1}] T^k \hat{h} + a_1 \frac{1}{n^{1/p}} T^{n-1} T\hat{h} - n a_n \frac{1}{n^{1/p}} T^{n-1} T^{n+1} \hat{h}.$$

The middle term in (3.9) tends to 0 a.e., since

$$\left| \frac{T^n \hat{h}}{n^{1/p}} \right|^p \le \frac{\mathbf{T}^n(|\hat{h}|^p)}{n} \to 0$$

by the ergodic theorem for $|\hat{h}|^p \in L_1$. This implies that the last term, of the order $T^{2n}\hat{h}/n^{1/p+\alpha}$, also converges to 0 a.e. Finally, for the sum in (3.9), note (again) that $k a_k - (k-1) a_{k-1} = -\alpha a_{k-1}$. Since $\hat{h} \in L_p$, the series $\sum_{j=1}^{\infty} a_j \mathbf{T}^j |\hat{h}|$ is a.e. absolutely convergent, to the function $h' = |\hat{h}| - (I - \mathbf{T})^{\alpha} |\hat{h}| \in L_p$. Hence the first term in (3.9) satisfies

$$\begin{aligned} \left| \frac{1}{n^{1/p}} T^{n-1} \sum_{k=2}^{n} [k a_k - (k-1) a_{k-1}] T^k \hat{h} \right| &= \left| \alpha \frac{1}{n^{1/p}} T^{n-1} \sum_{k=2}^{n} a_{k-1} T^k \hat{h} \right| \\ &\le \alpha \frac{1}{n^{1/p}} \mathbf{T}^n \sum_{k=1}^{n} a_k \mathbf{T}^k |\hat{h}| \\ &\le \alpha \frac{1}{n^{1/p}} \mathbf{T}^n h' \to 0. \end{aligned}$$

STEP IV: To prove that $A_n/n^{1-\alpha} + C_n/n^{1-\alpha} \to 0$ a.e. in case (iii), let $r = 1/1-\alpha$. Since $\alpha < 1/q$, we have $r < p$, so $g \in L_p$ is also in L_r, and by steps I and II (with p replaced by r and $\alpha = 1 - 1/r$) we have

$$A_n/n^{1-\alpha} + C_n/n^{1-\alpha} = A_n/n^{1/r} + C_n/n^{1/r} \to 0 \text{ a.e.}$$

STEP V: $B_n/n^{1-\alpha} \to 0$ a.e. in case (iii). As was shown at the beginning of Step III, the last term on the right-hand side of (3.6) converges to 0 a.e. (no restrictions on $\alpha \in (0,1)$ are needed).

CLAIM 2: *If $\alpha < 1/q$, then for every $h \in L_p$,*

$$\frac{1}{n^{1-\alpha}} T^{n-1} \sum_{k=1}^{n} k a_k T^k h \quad \text{converges a.e.} \tag{3.10}$$

Proof of Claim 2: Since $1-\alpha = 1/p + (1-\alpha q)/q$, Hölder's inequality yields

$$\left|\frac{1}{n^{1-\alpha}} T^{n-1} \sum_{k=1}^{n} k a_k T^{k+n-1} h\right| \le \left[\frac{1}{n^{1-\alpha q}} \sum_{k=1}^{n} (k a_k)^q\right]^{1/q} \left[\frac{1}{n} \sum_{k=n}^{2n-1} \mathbf{T}^k(|h|^p)\right]^{1/p}.$$

Since $\alpha q < 1$, we have

$$\frac{1}{n^{1-\alpha q}} \sum_{k=1}^{n} \frac{1}{k^{\alpha q}} \to \frac{1}{1-\alpha q}.$$

Using (2.2) (and $\tilde{h}$ defined in the proof of Claim 1), we obtain

$$\left|\frac{1}{n^{1-\alpha}} T^{n-1} \sum_{k=1}^{n} k a_k T^{k+n-1} h\right| \le C(3\tilde{h})^{1/p}.$$

By the Banach principle, it is now enough to prove (3.10) for h in a dense subspace.

To prove (3.10) for $h \in L_p$ invariant, replace α in (2.6) by $1-\alpha$ (and take $T = I$) to obtain

$$\frac{1}{n^{1-\alpha}} \sum_{k=1}^{n} k a_k^{(\alpha)} = K \frac{1}{n^{1-\alpha}} \sum_{k=2}^{n+1} \frac{1}{k^{\alpha}} + \frac{1}{n^{1-\alpha}} \sum_{k=1}^{n} \epsilon_k \to \frac{K}{1-\alpha}.$$

For $h = \hat{h} - T\hat{h}$ with $\hat{h} \in L_p$, we replace $n^{1/p}$ in (3.9) by $n^{1-\alpha}$, and the a.e. convergence to 0 follows from the estimates of the terms of (3.9), since $1-\alpha > 1/p$. This finishes the proof of Theorem 3.2. ∎

Remarks: 1. The conclusion of part (ii) of Theorem 3.2 can hold for f not in $(I-T)^{1/q} L_p$. Let $p = 2$, and take a sequence $\{f_n\}$ of i.i.d. random variables with mean 0 and variance 1, so the shift θ yields $f_n = f_0 \circ \theta^n$, with $f_0 \in L_2$. By independence, $\|(1/\sqrt{n}) \sum_{k=0}^{n-1} f_0 \circ \theta^k\|_2 = 1$, so by Corollary 2.15, $f_0 \notin (I-T)^{1/2} L_2$. However, the Law of the Iterated Logarithm [La, p. 49] implies that $(1/\sqrt{n \log n}) \sum_{k=0}^{n-1} f_k \to 0$ a.e. Note that by Theorem 2.17, $f_0 \in (I-T)^{\alpha} L_2$ for every $\alpha < \frac{1}{2}$.

2. In fact, the maximal function for (3.10) is in L_p: Let $p_1 < p$ have dual index $q_1 < 1/\alpha$, and replace p by p_1 in the proof of Claim 2. Since $|h|^{p_1} \in L_{p/p_1}$, the function $\tilde{h}$ (now defined with p_1) is also in L_{p/p_1}, which yields the result.

3. The method of proof of Step V yields also a short proof of the result of Irmisch, along the lines of Déniel's [Den] proof for probability preserving transformations (the new ingredient is the use of the inequality $|T^k g|^p \le \mathbf{T}^k(|g|^p)$ [Kr, p. 65]).

THEOREM 3.6: *Let T be a Dunford–Schwartz operator in $L_1(\mu)$ of a probability space, and fix $1 < p < \infty$, with dual index $q = p/(p-1)$. Let $0 < \alpha < 1$, and $f \in (I-T)^\alpha L_p$.*

(i) *If $\alpha > 1/q$, then the series $\sum_{k=1}^\infty T^k f/k^{1/p}$ converges a.e.*

(ii) *If $\alpha \le 1/q$, then for any $\epsilon > 0$, the series $\sum_{k=2}^\infty T^k f/k^{1-\alpha}(\log k)^{1+\epsilon}$ converges a.e.*

Proof: (i) By Theorem 3.2 (i), $S_n f/n^{1/p} \to 0$ a.e. Since $1/p + \alpha > 1$, we have a.e. convergence in (3.1) with $\gamma = 1/p$.

(ii) Fix $\epsilon > 0$, and denote $\delta = 1+\epsilon$, $\gamma = 1-\alpha$. By Abel's summation by parts

$$\sum_{k=2}^{n} \frac{T^k f}{k^{1-\alpha}(\log k)^{1+\epsilon}} =$$

$$\frac{S_n f}{n^\gamma (\log n)^\delta} - \frac{Tf}{2^\gamma(\log 2)^\delta} + \sum_{k=2}^{n-1}\Big[\frac{1}{k^\gamma(\log k)^\delta} - \frac{1}{(k+1)^\gamma(\log(k+1))^\delta}\Big] S_k f.$$

By Theorem 3.2 (ii)–(iii), $S_n f/[n^{1-\alpha}(\log n)^\alpha] \to 0$ a.e., so to prove the theorem it remains to prove a.e. convergence of the last series. By the mean value theorem,

$$\frac{1}{k^\gamma(\log k)^\delta} - \frac{1}{(k+1)^\gamma(\log(k+1))^\delta} \le \frac{\gamma\log(k+1)+\delta}{k^{\gamma+1}(\log k)^{\delta+1}} \le \frac{C}{k^{\gamma+1}(\log k)^\delta}.$$

This gives an a.e. estimation for the series in question by

$$C\sum_{k=2}^{n-1} \frac{1}{k(\log k)^\delta}\frac{|S_k f|}{k^{1-\alpha}}.$$

Since $f \in (I-T)^\alpha L_1$ and T is mean ergodic in L_1, $\|S_k f/k^{1-\alpha}\|_1 \to 0$. By Lebesgue's theorem and the convergence of $\sum_{k=2}^\infty 1/k(\log k)^\delta$ for $\delta > 1$, we have

$$\int \sum_{k=2}^{\infty} \frac{1}{k(\log k)^\delta}\frac{|S_k f|}{k^{1-\alpha}}d\mu = \sum_{k=2}^{\infty}\frac{1}{k(\log k)^\delta}\Big\|\frac{S_k f}{k^{1-\alpha}}\Big\|_1 < \infty.$$

Hence $\sum_{k=2}^{n-1}\frac{1}{k(\log k)^\delta}\frac{|S_k f|}{k^{1-\alpha}}$ converges a.e., which proves the theorem. ∎

COROLLARY 3.7: *Let T be a Dunford–Schwartz operator in $L_1(\mu)$ of a probability space, and fix $1 < p < \infty$. Let $f \in L_p$, and assume that for some $0 < \beta < 1$ we have $\sup_n \|(1/n^{1-\beta}) \sum_{k=1}^n T^k f\|_p < \infty$.*

(i) *If $\beta > 1 - 1/p$, then the series $\sum_{k=1}^\infty T^k f/k^{1/p}$ converges a.e., so $(1/n^{1/p}) \sum_{k=1}^n T^k f \to 0$ a.e.*

(ii) *If $\beta \le 1 - 1/p$, then for every $\gamma > 1 - \beta$, the series $\sum_{k=1}^\infty T^k f/k^\gamma$ converges a.e., so $(1/n^\gamma) \sum_{k=1}^n T^k f \to 0$ a.e.*

Proof: By Kronecker's lemma, the convergence of the series in each case implies the second convergence. By Theorem 2.17, for every $\alpha \in (0, \beta)$ we have $f \in (I - T)^\alpha L_p$.

(i) Take any α with $1/q < \alpha < \beta$, and apply Theorem 3.6(i).

(ii) Fix $\gamma > 1 - \beta$, and take α with $1 - \gamma < \alpha < \beta$. Then $\gamma > 1 - \alpha$, and apply Theorem 3.6(ii). ∎

Remarks: 1. The condition does not imply $f \in (I - T)^\beta L_p$ (e.g., $p = 2$, $\beta = \frac{1}{2}$, and $\{T^k f\}$ i.i.d.).

2. For an ergodic probability preserving transformation θ, Assani and Nicolaou [ANi, Theorems 5 and 6] obtained for Wiener–Wintner functions a.e. simultaneous convergence, for every λ with $|\lambda| = 1$, of $\sum_{k=1}^\infty \lambda^k f \circ \theta^k / k^\gamma$, under appropriate connections between γ and $p \ge 2$.

Gaposhkin [G-1], [G-2], [G-3] has used spectral methods for the study of pointwise ergodic theorems for unitary operators in L_2, and it is interesting to compare his results with ours, when T is induced by an invertible probability preserving transformation (and is then unitary in L_2).

1. In [G-3, Theorem 6a] (see also [G-1]), there is a sufficient condition for the a.e. (and L_2-norm) convergence of the left hand side of (3.1) (which then implies that $f \in (I - T)^{1-\gamma} L_2$). In Theorem 3.6 (for T as above), with $p = q = 2$, when $\alpha \le \frac{1}{2}$, we can obtain from [G-3] only the a.e. convergence of $\sum_{k=1}^n T^k f/k^\gamma$ for any $\gamma > 1 - \alpha$, which is weaker than Theorem 3.6(ii). When $\alpha > \frac{1}{2}$, we have $f \in (I - T)^{\alpha'} L_2$ for $\alpha' \le \frac{1}{2}$, but then [G-3] yields only the a.e. convergence of $\sum_{k=1}^n T^k f/k^\gamma$ for $\gamma > \frac{1}{2}$, so also in this case Theorem 3.6(i) gives a better result.

2. The following "rates of a.e. convergence", for T unitary in L_2 and $f \in L_2$ with $\sup_n \|(1/n^{1-\beta}) \sum_{k=1}^n T^k f\|_2 < \infty$, are special cases of Gaposhkin's [G-2, Theorem 3], cases (vii), (iv), and (iii), respectively:

(a) If $\beta > \frac{1}{2}$, then

$$\frac{1}{n^{1/2}(\log n\ \log\log n)^{1/2}(\log\log\log n)^{(1+\epsilon)/2}}\sum_{k=1}^{n} T^k f \to 0.$$

(b) If $\beta = \frac{1}{2}$, then

$$\frac{1}{n^{1/2}(\log n)^{3/2}(\log\log n)^{(1+\epsilon)/2}}\sum_{k=1}^{n} T^k f \to 0.$$

(c) If $\beta < \frac{1}{2}$, then

$$\frac{1}{n^{1-\beta}(\log n)^{1/2}(\log\log n)^{(1+\epsilon)/2}}\sum_{k=1}^{n} T^k f \to 0.$$

For T induced by an invertible probability preserving transformation, Corollary 3.7 (with $p=2$) gives a better result for $\beta > \frac{1}{2}$, while [G-2] yields a better result for $\beta \le \frac{1}{2}$. On the other hand, for $f \in (I-T)^{\beta} L_2$, Theorem 3.2 yields better rates in all the cases.

The question whether $(1/n^{1/p})\sum_{k=0}^{n-1} T^k f$ converges a.e. for every $f \in (I-T)^{1/q} L_p$ has a negative answer in general, as shown in part (ii) of the next proposition, and the counter-example is for T induced by a probability preserving transformation.

Fix $1 < p < \infty$, and put $\alpha = 1/q$. For $r > p$ we have $\alpha < 1 - 1/r$, so Theorem 3.2(iii) yields a.e. convergence (to 0) of $(1/n^{1/p})\sum_{k=0}^{n-1} T^k f$ for $f \in (I-T)^{\alpha} L_r \subset (I-T)^{\alpha} L_p$. For T induced by a probability preserving transformation of ([0,1],dx) we use the techniques of [BDenDe] to find a larger subspace $L' \subset L_p$, such that $(1/n^{1/p})\sum_{k=0}^{n-1} T^k f$ converges a.e. for $f \in (I-T)^{\alpha} L'$. For a function g defined on [0,1], let $g^{\#}$ be the decreasing rearrangement of $|g|$, and define the Lorentz space

$$L(p,1) := \left\{ g\colon \int_0^1 x^{1/p} g^{\#}(x)\frac{dx}{x} < \infty \right\}.$$

We then have $\bigcup_{r>p} L_r(dx) \subsetneq L(p,1) \subsetneq L_p(dx)$.

PROPOSITION 3.8: (i) *Let T be induced on $L_p([0,1],dx)$ by an ergodic transformation of [0,1] which preserves Lebesgue measure. If $f \in (I-T)^{1/q} L(p,1)$, then $(1/n^{1/p})\sum_{k=0}^{n-1} T^k f$ converges to 0 a.e.*

(ii) *There exists T induced on $L_2([0,1],dx)$ by an ergodic transformation of [0,1] which preserves Lebesgue measure, and there exists $f \in (I-T)^{\frac{1}{2}} L_2(dx)$, such that $(1/\sqrt{n})\sum_{k=0}^{n-1} T^k f$ is a.e. non-convergent.*

Proof: (i) Let $f \in (I-T)^{1/q}L_p$. By Corollary 2.15, $(1/n^{1/p})\sum_{k=0}^{n-1}T^kf$ converges to 0 in L_p-norm, so if it converges a.e., the limit is 0 a.e. Write $f=(I-T)^{1/q}g$, with $g\in L_p$ and $\int g\,dm=0$ (so $g\in\overline{(I-T)L_p}$ by ergodicity), and take the decomposition $\sum_{k=0}^{n-1}T^kf = A_n+B_n+C_n$ given by (3.2)–(3.4) (with $\alpha=1/q$). In Steps I and II of the the proof of Theorem 3.2 it was shown that $A_n/n^{1/p}\to 0$ and $C_n/n^{1/p}\to 0$ a.e. By (3.6) (adding the same last term to each sum),

$$\frac{\alpha B_n}{n^{1/p}} = -T^{n-1}\sum_{j=1}^{n+1}\frac{1}{n^{1/p}}ja_jT^jg+\frac{1}{n^{1/p}}(n+1)a_{n+1}\sum_{k=n}^{2n}T^kg.$$

By the ergodic theorem, $(1/n)\sum_{k=n}^{2n}T^kg\to\int g\,dm=0$ a.e., so (2.2) yields a.e. convergence to 0 of the last term above. Since $ja_j^{(\alpha)}=\alpha A_{j-1}^{\beta-1}$ for $j\geq 1$, where $\{A_k^\gamma\}$ are the Cesàro coefficients and $\beta=1-\alpha=1/p$, (2.2) yields that a.e. convergence of $B_n/n^{1/p}$ is equivalent to a.e. convergence of $(1/A_n^\beta)\sum_{j=n}^{2n}A_{j-n}^{\beta-1}T^jg$.

This type of "weighted averages" for measure preserving transformations was studied in [BDenDe]. It was proved there that a.e. convergence holds for weighted averages as soon as it holds for the sequence of averages with the same weights rearranged in increasing order. In our problem, the j-th given weight is 0 for $0\leq j<n$ and $A_{n-j}^{\beta-1}/A_n^\beta$ for $n\leq j\leq 2n$. When these weights are put in increasing order, we have for $n\leq j\leq 2n$ the weight $A_{2n-j}^{\beta-1}/A_n^\beta$ (and 0 for $j<n$). The rearranged weights behave like the weights of the Cesàro means of order β. It was proved in [BDenDe] that for $0<\beta<1$ the Cesàro means of order β converge a.e. for any function in the Lorentz space $L(1/\beta,1)=L(p,1)$ (and this space is optimal in terms of integrability conditions).

(ii) Let T be the ergodic Lebesgue measure preserving transformation of [0,1] defined by the stacking method with division by 2. Let $g^+\in L_2$ be non-negative, with $g^+\notin L(2,1)$. By an adaptation of the construction in p. 709 of [BDenDe], there exists another transformation which preserves Lebesgue measure, with induced operator S, such that

$$\liminf_n\frac{1}{\sqrt{n}}\sum_{j=n}^{2n-1}\frac{T^k(Sg^+)}{\sqrt{j-n+1}}<\limsup_n\frac{1}{\sqrt{n}}\sum_{j=n}^{2n-1}\frac{T^k(Sg^+)}{\sqrt{j-n+1}}=+\infty\quad\text{a.e.}\tag{3.11}$$

Let $g^-=\int g\,dx$. Since g^- is constant, $(1/\sqrt{n})\sum_{j=n}^{2n-1}T^k(Sg^-)/\sqrt{j-n+1}$ converges a.e. Define $g=g^+-g^-$. Then $\int Sg\,dx=0$, so by ergodicity $Sg\in\overline{(I-T)L_2}$. For $f=(I-T)^{1/2}Sg$, (3.11) yields that

$$\limsup_n\frac{1}{\sqrt{n}}\sum_{k=1}^{n}T^kf=+\infty\quad\text{a.e.}\qquad\blacksquare$$

Example: T induced by a probability preserving transformation,

$$\left\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} T^k f \right\|_2 \to 0,$$

but $f \notin (I-T)^{1/2} L_2$.

This example is inspired by Lacey's work [L]. Let β be an irrational number in [0,1], and let T be induced in $L_2[0,1]$ by the measure preserving transformation $\theta(\xi) = \xi + \beta \bmod 1$. The desired function $f \in L_2$ will be defined by its Fourier coefficients. For $j \geq 3$ we define the half open interval

$$I_j = \left[\frac{\log(1+j)}{1+j}, \frac{\log j}{j} \right[.$$

The intervals I_j are disjoint, with $\bigcup_{j\geq 3} I_j =]0, (\log 3)/3[$. Denote the fractional part of any real r by $\{r\}$, and let $k_j = \inf\{k > 0 : \theta^k(0) = \{k\beta\} \in I_j\}$ (the first entrance time of 0 to I_j). We define $c_{k_j} = 1/j$, and for non-negative $k \notin \{k_j\}$ we define $c_k = 0$. For negative values put $c_{-k} = c_k$. Clearly $\sum_{k\in\mathbf{Z}} c_k^2 < \infty$, so there is an $f \in L_2[0,1]$ defined by $f(t) = \sum_{k\in\mathbf{Z}} c_k e^{2\pi ikt}$ (with convergence of the series in L_2). Solving the fractional Poisson equation by Fourier series shows that $f \in (I-T)^{1/2} L_2$ is equivalent to

$$\sum_{k\in\mathbf{Z}} \frac{|c_k|^2}{|1 - e^{2\pi ik\beta}|} < \infty.$$

Since $|1 - e^{2\pi it}| < 2\pi t$ and $\{k_j\beta\} < (\log j)/j$, our definition of c_k yields

$$\sum_{k\in\mathbf{Z}} \frac{|c_k|^2}{|1 - e^{2\pi ik\beta}|} = \sum_{j\geq 3} \frac{|c_{k_j}|^2}{|1 - e^{2\pi ik_j\beta}|} \geq \sum_{j\geq 3} \frac{1}{j^2}\{k_j\beta\} \geq \sum_{j\geq 3} \frac{1}{j\log j} = \infty.$$

Hence $f \notin (I-T)^{1/2} L_2$. The proof that our f satisfies

$$\lim_n \frac{1}{n} \left\| \sum_{k=0}^{n-1} T^k f \right\|_2^2 = 0$$

is rather delicate, and uses arguments similar to Lacey's [L]. The main point in the required analysis is the proof of

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=m^b}^{m^a} \frac{1}{j^2} \left| \frac{1 - e^{2\pi imk_j\beta}}{1 - e^{2\pi ik_j\beta}} \right|^2 = 0 \quad \text{for arbitrary } b < 1 < a.$$

Remark: An example of a positive Dunford–Schwartz contraction T (which is not Markovian), with the same properties as the previous example, is given in the next section.

Another important class of DS operators is that of Harris recurrent Markov operators (Harris operators) with a finite invariant measure (see [F] or [Kr]), which generalize positive recurrent Markov matrices. This class includes the Markov operators satisfying Doeblin's condition. Horowitz [Ho] proved that an ergodic Harris operator P with invariant probability has a positive integer L (the period of P), such that for every $f \in L_1$ the sequence $\{P^{nL}f\}$ converges a.e.

THEOREM 3.9: *Let P be a Harris recurrent operator with equivalent invariant probability. If $f \in (I-P)^\alpha L_1$ for some $0 < \alpha < 1$, then the series $\sum_{k=1}^\infty P^k f/k^{1-\alpha}$ converges a.e., and, consequently, $(1/n^{1-\alpha})\sum_{k=1}^n P^k f \to 0$ a.e.*

Proof: The invariant σ-algebra of a Harris operator is atomic [F]. By applying Horowitz's theorem on each atom we obtain that $g^* := \sup_n |P^n g|$ is finite a.e., for every $g \in L_1$. Let $g \in L_1$. By Proposition 2.4(i),

$$\begin{aligned}
&\left|\sum_{j=0}^n (j+1)a_{j+1}^{(1-\alpha)} P^j (I-P)^\alpha g\right| \\
&\le (1-\alpha)|g| + \left|\sum_{k=n+1}^\infty \left(\sum_{j=0}^n (j+1)a_{j+1}^{(1-\alpha)} a_{k-j}^{(\alpha)}\right) P^k g\right| \\
&\le (1-\alpha)|g| + \sum_{k=n+1}^\infty \left(\sum_{j=0}^n (j+1)a_{j+1}^{(1-\alpha)} a_{k-j}^{(\alpha)}\right) g^*.
\end{aligned}$$

By Lemma 2.3 we have

$$\sup_n \left|\sum_{j=0}^n (j+1)a_{j+1}^{(1-\alpha)} P^j (I-P)^\alpha g\right| \le 2(1-\alpha)g^*.$$

By Banach's principle, to get a.e. convergence of

$$\sum_{j=0}^n (j+1)a_{j+1}^{(1-\alpha)} P^j (I-P)^\alpha g \quad \text{for every } g \in L_1,$$

it is enough to prove it for g invariant and for $g \in (I-P)L_1$ (since P is mean ergodic in L_1). For g invariant, $(I-P)^\alpha g = 0$; for $g = (I-P)h$ with $h \in L_1$, the convergence follows from the computations in Proposition 2.6, with the norm estimation using $||x||$ replaced by pointwise estimation using h^*. For $f \in$

$(I-P)^\alpha L_1$ we now have a.e. convergence of $\sum_{j=0}^n (j+1)a_{j+1}^{(1-\alpha)}P^j f$, which is equivalent to the assertion of the theorem. ∎

A special class of Markov operators consists of those corresponding to reversible Markov chains. In terms of the transition operator P, it means that P has an invariant probability, and in L_2 we have $P^* = P$. Note that for any DS operator T, also its dual T^* is a DS operator.

THEOREM 3.10: *Let T be a Dunford–Schwartz operator in a probability space, with $T^* = T$. Then*

(i) *For $1 < p < \infty$ and $0 < \alpha < 1$ there is a constant $C_{\alpha,p}$ such that*

$$\left\| \sup_{n\geq 1} \left| \sum_{j=0}^n (j+1)a_{j+1}^{(1-\alpha)} T^j (I-T)^\alpha g \right| \right\|_p \leq C_{\alpha,p} \|g\|_p \quad \text{for every } g \in L_p.$$

(ii) *For $1 < p < \infty$, $0 < \alpha < 1$, and $f \in (I-T)^\alpha L_p$, the series $\sum_{k=1}^\infty T^k f/k^{1-\alpha}$ converges a.e. and, consequently, $(1/n^{1-\alpha})\sum_{k=1}^n T^k f \to 0$ a.e.*

Proof: By Stein's theorem [St], $g^* := \sup_n |T^n g|$ is in L_p for every $g \in L_p$, and there exists a constant C_p such that $\|g^*\|_p \leq C_p \|g\|_p$. (i) is now proved along the lines of the previous proof, and so is (ii). ∎

THEOREM 3.11: *Let T be a self-adjoint positive contraction of $L_2(\mu)$, and let $0 < \alpha < 1$. Then:*

(i) *There exists a constant C_α such that*

$$\left\| \sup_{n\geq 1} \left| \sum_{j=0}^n (j+1)a_{j+1}^{(1-\alpha)} T^j (I-T)^\alpha g \right| \right\|_2 \leq C_\alpha \|g\|_2 \quad \text{for every } g \in L_2.$$

(ii) *For $f \in (I-T)^\alpha L_2$, the series $\sum_{k=1}^\infty T^k f/k^{1-\alpha}$ converges a.e. and, consequently, $(1/n^{1-\alpha})\sum_{k=1}^n T^k f \to 0$ a.e.*

The proof is the same as that of the previous theorem, except that we use the appropriate version of Stein's theorem [Kr, p. 190], which uses Akcoglu's theorem instead of the Dunford–Schwartz theorem.

THEOREM 3.12: *Let $1 < p < \infty$, and let T be a positive contraction of $L_p(\mu)$. If $f \in (I-T)^\alpha L_p$ for some $0 < \alpha < 1$, then the series $\sum_{k=1}^\infty T^k f/k$ converges a.e.*

Proof: As in the proof of Akcoglu's theorem (see [Kr, p. 189]), we may assume that μ is a probability. Let $f \in (I-T)^\alpha L_p$. Then $f \in \overline{(I-T)L_p}$, and by

Akcoglu's theorem, $S_kf/k \to 0$ a.e. (where $S_kf = \sum_{j=1}^k T^j f$). Since μ is a probability,

$$\left\| \frac{S_n f}{n^{1-\alpha}} \right\|_1 \le \left\| \frac{S_n f}{n^{1-\alpha}} \right\|_p,$$

which tends to 0 by Theorem 2.15. The theorem now follows from the proof of Theorem 3.1. ∎

THEOREM 3.13: *Let T be a contraction of $L_1(\mu)$ of a σ-finite measure space, $0 < \alpha < 1$, and $f \in L_1(\mu)$. Then $f \in (I-T)^\alpha L_1(\mu)$ if and only if*

$$\sup_N \left\| \sum_{j=0}^N (j+1) a_{j+1}^{(1-\alpha)} T^j f \right\|_1 < \infty.$$

Proof: When $f \in (I-T)^\alpha L_1$, we obtain the boundedness condition from Proposition 2.4(ii).

Assume now $\sup_N \| \sum_{j=0}^N (j+1) a_{j+1}^{(1-\alpha)} T^j f \|_1 < \infty$. We identify $L_1(\mu)$ with the space $M(\mu)$ of finite signed measures absolutely continuous with respect to μ, and let $d\nu = f d\mu$. Since $L_\infty(\mu)^*$ is the space of finitely additive measures absolutely continuous with respect to μ, an application of Theorem 2.13 to T^{**} yields a finitely additive ρ such that $(I - T^{**})^\alpha \rho = \nu$. We now adapt the argument of [LiSi]. Let $\rho = \eta + \rho_0$ be the Hewitt–Yosida decomposition of ρ, with $\eta \in M(\mu)$ and ρ_0 a pure charge (i.e., $|\rho_0|$ does not dominate a countably additive measure). Then

$$\nu = (I - T^{**})^\alpha \rho = (I - T_\alpha^{**})\rho = (I - T_\alpha^{**})\eta + \rho_0 - T_\alpha^{**}\rho_0.$$

Since $T_\alpha^{**}\eta = T_\alpha \eta \in M(\mu)$, we have that $(I - T_\alpha^{**})\eta$ is countably additive. $\|T_\alpha\| \le 1$ now yields

$$\|\rho_0\| \ge \|T_\alpha^{**}\rho_0\| = \|(I - T_\alpha)\eta - \nu + \rho_0\| = \|(I - T_\alpha)\eta - \nu\| + \|\rho_0\|.$$

Hence $\nu = (I - T_\alpha)\eta = (I-T)^\alpha \eta$. ∎

Remark: Theorem 2.13 gives the equivalence of the above conditions for dual operators in dual spaces. However, even for T Dunford–Schwartz, it does not imply directly the previous theorem.

4. Normal contractions in Hilbert spaces

In this section we look at special properties and characterizations of fractional coboundaries of normal contractions in Hilbert spaces. In complex Hilbert spaces, the spectral theorem is the main tool for the study.

PROPOSITION 4.1: *Let T be a normal contraction in a (real or complex) Hilbert space H. Then for $0 < \alpha \le 1$ we have $(I - T^*)^\alpha H = (I - T)^\alpha H$.*

Proof: We first prove for $\alpha = 1$. Since T is normal, for $y \in H$ and every N we have

$$\left\| \sum_{k=0}^{N} T^k y \right\|^2 = \left\| \sum_{k=0}^{N} T^{*k} y \right\|^2 . \tag{4.1}$$

By [BuW] (see also [LiSi]), $y \in (I - T)H$ if and only if $\sup_N \|\sum_{k=0}^N T^k y\| < \infty$. Applying this also to T^* and using (4.1), we obtain $(I - T)H = (I - T^*)H$. Since $(I - T)^\alpha = I - T_\alpha$ and T_α is also a normal contraction, the proposition is proved. ∎

Remark: A. Iwanik suggested to deduce Proposition 4.1 from the following general result (which he proved for H complex, using the spectral theorem). In fact, we obtain $(I - T)^r H = (I - T^*)^r H$ for every $r \ge 0$.

PROPOSITION 4.2: *Let S be a normal operator in a (real or complex) Hilbert space H. Then $SH = S^*H$.*

Proof: The operator $R = \sqrt{SS^*}$ is well-defined even in the real case (as a strong limit of polynomials in SS^* with real coefficients), and is symmetric (i.e., $R^* = R$). By [RSz-N, p. 283] (also in the real case), there is a unitary operator U commuting with R such that $S = RU = UR$. Then $S = UR = U^*U^2R = S^*U^2$, which shows that S and S^* have the same range. ∎

Remark: Proposition 4.1 fails if T is not normal. Let $H = \ell_2$ and T be the shift $T(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots)$. Then $T^*(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots)$. The unit vector $e_1 = (1, 0, 0, \ldots)$ satisfies $T^*e_1 = 0$, so for $0 < \alpha < 1$ we have $(I - T^*)^\alpha e_1 = e_1$. Since $T^j e_1 = e_{j+1}$ for $j \ge 1$, we have

$$\left\| \sum_{j=1}^{N} \frac{T^j e_1}{j^{1-\alpha}} \right\|^2 = \sum_{j=1}^{N} \frac{1}{j^{2(1-\alpha)}} .$$

For $\alpha \ge \frac{1}{2}$, the divergence of the series and Theorem 2.13 yield $e_1 \notin (I - T)^\alpha H$.

PROPOSITION 4.3: *Let T be an isometry of a (real or complex) Hilbert space H. Then $(I - T)H \subset (I - T^*)H$, and equality holds if and only if T is invertible.*

Proof: $T^*T = I$ implies $I - T = T^*T - T = (T^* - I)T$, which yields the desired inclusion. If T is invertible, T^* is also an isometry, and equality holds.

Assume that T is not invertible; so there is $z \in H$ with $TT^*z \neq z$. Since T and T^* have the same fixed points, the ergodic decomposition shows that we may take $z \in \overline{(I-T)H}$. Put $y = (I - T^*)z$. If $y = -(I-T)x$, we may assume $x \in \overline{(I-T)H}$. We have $Tx = x+y$, so $(I-T^*)(x+y) = (I-T^*)Tx = (T-I)x = y = (I-T^*)z$. Hence $(I-T^*)(x+y-z) = 0$ for $x+y-z \in \overline{(I-T)H}$, so $z = x+y$, yielding $z = Tx$. Hence $TT^*z = TT^*Tx = Tx = z$ — a contradiction. ∎

THEOREM 4.4: *Let T be a normal contraction in a complex Hilbert space, with spectral measure $E(.)$ on the Borel subsets of the spectrum $\sigma(T)$. Let $0 < \alpha < 1$ and $y \in H$. Then $y \in (I-T)^\alpha H$ if and only if*

$$\int_{\sigma(T)} \frac{1}{|1-\lambda|^{2\alpha}} \langle E(d\lambda)y, y\rangle < \infty. \tag{4.2}$$

Proof: Denote $\mu(.) = \langle E(.)y, y\rangle$. Assume that (4.2) holds. In particular $\mu(\{1\}) = 0$. Let $\phi(\lambda) = \sum_{j=1}^\infty a_j^{(\alpha)}\lambda^j$. Then $|1-\lambda|^{2\alpha} = |1-\phi(\lambda)|^2$. Since $T_\alpha = \sum_{j=1}^\infty a_j^{(\alpha)} T^j$, for any $N \geq 1$ the spectral theorem yields

$$\begin{aligned}\left\|\sum_{k=0}^N T_\alpha^k\right\|^2 &= \int_{\sigma(T)} \left|\sum_{k=0}^N \phi(\lambda)^k\right|^2 d\mu \\ &= \int_{\sigma(T)} \frac{|1-\phi(\lambda)^{N+1}|^2}{|1-\phi(\lambda)|^2} d\mu \leq \int_{\sigma(T)} \frac{4}{|1-\lambda|^{2\alpha}} d\mu.\end{aligned}$$

(4.2) yields $\sup_N \|\sum_{k=0}^N T_\alpha^k y\| < \infty$, and $y \in (I-T_\alpha)H = (I-T)^\alpha H$ by [BuW].

Assume now that $y = (I-T)^\alpha x$, and we may take $x \in \overline{(I-T_\alpha)H}$. By the spectral theorem, for any Borel set A we have

$$\begin{aligned}&\langle E(A)y, y\rangle = \langle E(A)(I-T_\alpha)x, (I-T_\alpha)x\rangle = \langle E(A)(I-T_\alpha)(I-T_\alpha^*)x, x\rangle \\ &= \int_A (1 - \sum_{j=1}^\infty a_j^{(\alpha)}\lambda^j)(1 - \sum_{k=1}^\infty \bar{a}_k^{(\alpha)}\lambda^k)\langle E(d\lambda)x, x\rangle = \int_A |1-\lambda|^{2\alpha}\langle E(d\lambda)x, x\rangle.\end{aligned}$$

This shows that the measure $\langle E(.)y, y\rangle$ is absolutely continuous with respect to $\langle E(.)x, x\rangle$, with Radon–Nikodym derivative $|1-\lambda|^{2\alpha}$. Since $x \in \overline{(I-T)H}$, we have $\langle E(\{1\}x, x\rangle = 0$, so the measures are equivalent. (4.2) follows from

$$\int_{\sigma(T)} \frac{1}{|1-\lambda|^{2\alpha}}\langle E(d\lambda)y, y\rangle = \int_{\sigma(T)} \langle E(d\lambda)x, x\rangle = \|x\|^2. \qquad ∎$$

Remark: For T symmetric and $\alpha = \frac{1}{2}$, the result is implicit in [KiV].

The next example shows that the converse of Corollary 2.15 may fail even for symmetric contractions.

Example: A symmetric contraction T, $y \notin (I-T)^{\frac{1}{2}}H$ with $\|(1/\sqrt{n})\sum_{k=0}^{n-1} T^k y\| \to 0$.

Construction: Let $H = L_2[0,1]$ (with Lebesgue's measure), and define $Tf(t) = tf(t)$. Then T is a symmetric contraction, and $(I-T)^{\frac{1}{2}}f(t) = \sqrt{1-t}f(t)$. Thus, $f \in (I-T)^{1/2}H$ if and only if

$$\int_0^1 \frac{|f(t)|^2}{1-t}dt < \infty.$$

Clearly,

$$\left\|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} T^k f\right\|^2 = \frac{1}{n}\int_0^1 \Big(\sum_{k=0}^{n-1} t^k\Big)^2 |f(t)|^2 dt.$$

Fix $0 < a < 1$, and define $h(t)$ on $[0,1]$ by $h(t) = 0$ on $[0,a)$ and $h(t) = -1/\log(1-t)$ on $[a,1)$. Since $\lim_{t\to 1^-} h(t) = 0$, we define $h(1) = 0$. Then h is strictly decreasing and continuous on $[a,1]$, so $\int_0^1 h(t)dt < \infty$. Put $g(t) = \sqrt{h(t)}$. Then $g \notin (I-T)^{1/2}H$, since

$$\int_0^1 \frac{|g(t)|^2}{1-t}dt = \int_a^1 \frac{-1}{(1-t)\log(1-t)}dt = \log\big(-\log(1-t)\big)\Big|_a^1 = \infty.$$

We now assume $a > 1 - 1/e$, and show that $\|(1/\sqrt{n})\sum_{k=0}^{n-1} T^k g\| \to 0$. For $n > 1/(1-a)$ we have

$$\frac{1}{n}\int_0^1 \left(\sum_{k=0}^{n-1} t^k\right)^2 |g(t)|^2 dt = \tag{4.3}$$

$$\frac{-1}{n}\int_a^{1-1/n} \frac{(1-t^n)^2}{(1-t)^2\log(1-t)}dt - \frac{1}{n}\int_{1-1/n}^1 \frac{(\sum_{k=0}^{n-1} t^k)^2}{\log(1-t)}dt.$$

Since $(1/n)\big(\sum_{k=0}^{n-1} t^k\big)^2 \le n$ on $[0,1]$, the monotonicity of $h(t)$ on $[a,1]$ yields

$$-\frac{1}{n}\int_{1-1/n}^1 \frac{(\sum_{k=0}^{n-1} t^k)^2}{\log(1-t)}dt \le \frac{1}{1/n}\int_{1-1/n}^1 h(t)dt \le h(1-1/n) \underset{n\to\infty}{\longrightarrow} h(1) = 0.$$

The first integral on the right hand side of (4.3) clearly satisfies

$$\frac{-1}{n}\int_a^{1-1/n} \frac{(1-t^n)^2}{(1-t)^2\log(1-t)}dt \le \frac{-1}{n}\int_a^{1-1/n} \frac{1}{(1-t)^2\log(1-t)}dt.$$

Integrating by parts, we obtain, using $a > 1 - 1/e$ and $h(a) < 1$,

$$\frac{-1}{n}\int_a^{1-1/n} \frac{1}{(1-t)^2\log(1-t)}dt \le \frac{1}{1-h(a)}\left[\frac{1}{\log n} + \frac{1}{n(1-a)\log(1-a)}\right] \underset{n\to\infty}{\longrightarrow} 0.$$

This proves convergence to 0 of the first integral on the right hand side of (4.3).

Gordin and Lifshits (in Corollary IV.7.1 of [BoI]) proved the central limit theorem for a normal Markov operator T when $f \in L_2$ satisfies the following two conditions:

$$\liminf_{n\to\infty}\left[\left\|\sum_{k=0}^{n-1} T^k f\right\|^2 - \left\|\sum_{k=1}^{n} T^k f\right\|^2\right] < \infty \tag{4.4}$$

and

$$\lim_{n\to\infty}\left\|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} T^k f\right\| = 0. \tag{4.5}$$

It is shown there that in terms of the spectral measure E, for any normal operator (4.4) is equivalent to

$$\int_{\sigma(T)} \frac{1-|\lambda|^2}{|1-\lambda|^2}\langle E(d\lambda)f, f\rangle < \infty. \tag{4.6}$$

By Theorem 4.4 and Corollary 2.15, (4.6) and (4.5) are satisfied when $f \in \sqrt{I-T}H$.

Condition (4.6) is always satisfied by T unitary, so does not imply (4.5). For T symmetric, (4.6) holds if and only if $f \in \sqrt{I-T}H$, so the previous example shows that (4.5) does not imply (4.6). Finally, the example of the previous section shows that (4.5) and (4.6) together do not imply $f \in \sqrt{I-T}H$.

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